Conservation of divergence or curl type constraints of hyperbolic systems with the discontinuous Galerkin method

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Introduction

Example of divergence or curl constraint

- Consider one equation of a hyperbolic system where $\mathbf{u}$ is a vector

$$\partial_t \mathbf{u} + \boldsymbol{\nabla} \cdot \overline{\mathbf{G}} = 0$$

- If $\overline{\mathbf{G}}$ is an homothety, then

$$\partial_t \mathbf{u} + \nabla g = 0 \qquad \Longrightarrow \qquad \partial_t \left(\nabla \times \mathbf{u} \right) = 0.$$

  - e.g. wave system, low Mach number limit

- If $\overline{\mathbf{G}}$ is skew-symmetric, then

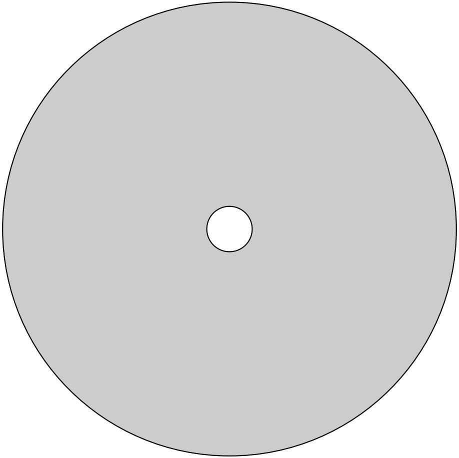
$$\partial_t \mathbf{u} + \nabla \times \mathbf{g} = 0 \qquad \Longrightarrow \qquad \partial_t \left(\nabla \cdot \mathbf{u} \right) = 0.$$

  - e.g. Maxwell system, (MHD equations)

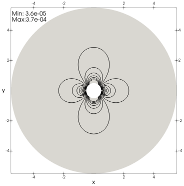
- Lebedev, 1964 MAC scheme, Yee, 1966 Yee scheme
- Formalized through differential geometry (de-Rham complex)
  - Electromagnetism
    - A. Bossavit : electromagnetism
    - R. Hiptmair : *Num. Math. 2001, Acta Num. 2002*
  - Finite Element Exterior Calculus
    - D. Arnold, *Acta Num. 2006, Bull. AMS 2010, SIAM 2018*
  - Extensions to polygonal/polyhedral meshes (VEM, HHO,CDO)
- A lot of applications also on hyperbolic systems (Staggered schemes)
  - MHD : preservation of $\nabla \cdot \mathbf{B} = 0$
  - Low Mach number flows (curl preservation)
  - GRP multiphase model : preservation of $\nabla \times \mathbf{w} = 0$

Inflow :

$$\begin{cases} \rho_\infty = 1 \\ \mathbf{u}^x_\infty = M_\infty \times c(\rho_\infty) \\ \mathbf{u}^y_\infty = 0 \end{cases} \Rightarrow$$



- Low Mach number flow, Roe scheme



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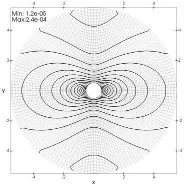
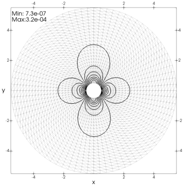
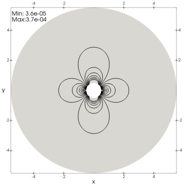


Quadrangle



Triangle

- Low Mach number flow, Roe scheme



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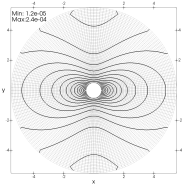

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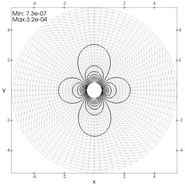
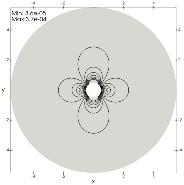


Triangle

- Low Mach number flow, Lax-Friedrich



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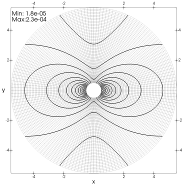

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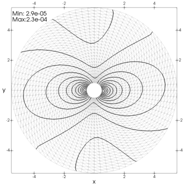
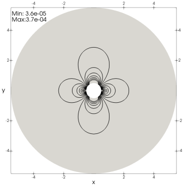


Triangle

Introduction

Low Mach number flows/Wave system

- Low Mach number flow, Roe scheme



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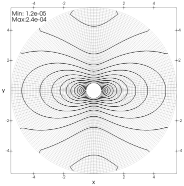
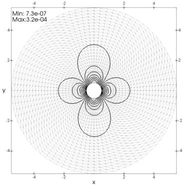
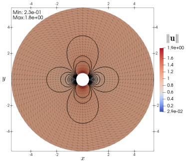


Quadrangle

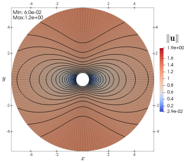


Triangle

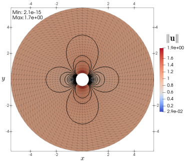
- Waves, Godunov scheme, long time limit



Exact



Quadrangle



Triangle

- A surprising result : Roe scheme is low Mach number accurate on triangles !
  - F.Rieper, G. Bader, *JCP 2009* : structured triangles
  - H. Guillard, *Comp. & Fluids 2009* : unstructured triangles/tetrahedra
  - P. Omnes, S. Dellacherie, F. Rieper *JCP 2009* : discrete Hodge-Helmholtz decomposition
  - J. Jung, VP, *SIAM SISC 2024* : high order DG on triangles and tetrahedra.
- Long time behaviour of the wave system (J. Jung, VP, *JCP 2022*)

$$\begin{cases} \partial_t \rho + \mathrm{div}_{\mathbf{x}} \mathbf{u} = 0 \\ \partial_t \mathbf{u} + \nabla P = 0 \end{cases}$$

$\Longleftrightarrow$ Conservation of the vorticity.

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$$\left\{\begin{array}{l}\partial_t\rho+\mathrm{div}_{\mathbf{x}}\mathbf{u}=0\\ \partial_t\mathbf{u}+\nabla P=0\end{array}\right.$$

$\Longleftrightarrow$ Conservation of the vorticity.

- Is there a similar framework for discontinuous approximation space as for staggered approximation ?

Outline of the talk

1. Focus on the low order triangular case
2. Continuous and discrete conforming de-Rham complex
3. Nonconforming discrete de-Rham complex and preservation of curl or divergence constraints
4. Numerical results

Focus on the low order triangular case

The continuous case (1/2)

Hodge-Helmholtz decomposition

Hodge-Helmholtz decomposition

$$\mathbf{u} = \mathbf{u}_\varphi + \mathbf{u}_{\boldsymbol{\Psi}} \qquad \text{with} \qquad \begin{cases} \mathbf{curl}_{\mathbf{x}} \mathbf{u}_\varphi = 0 \\ \mathrm{div}_{\mathbf{x}} \mathbf{u}_{\boldsymbol{\Psi}} = 0 \end{cases}$$

- When the domain is connected

$$\begin{cases} \mathbf{u}_\varphi = \nabla_{\mathbf{x}} \varphi \\ \mathbf{u}_{\boldsymbol{\Psi}} = \mathbf{curl}_{\mathbf{x}} \boldsymbol{\Psi} \end{cases}$$

- Usually computed by potentials extraction

$$\mathbf{u} = \mathbf{u}_\varphi + \mathbf{u}_{\boldsymbol{\Psi}}$$

$$\mathrm{div}_{\mathbf{x}} \mathbf{u} = \mathrm{div}_{\mathbf{x}} \left(\mathbf{u}_\varphi\right) + \underbrace{\mathrm{div}_{\mathbf{x}} \left(\mathbf{u}_{\boldsymbol{\Psi}}\right)}_{=0} = \Delta \varphi$$

Hodge-Helmholtz decomposition

$$\mathbf{u} = \mathbf{u}_\varphi + \mathbf{u}_\mathbf{\Psi} \qquad \text{with} \qquad \left\{ \begin{array}{l} \mathbf{curl}_\mathbf{x}\mathbf{u}_\varphi = 0 \\ \mathrm{div}_\mathbf{x}\mathbf{u}_\mathbf{\Psi} = 0 \end{array} \right.$$

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- Usually computed by potentials extraction

$$\mathbf{u} = \mathbf{u}_\varphi + \mathbf{u}_\mathbf{\Psi}$$

$$\mathbf{curl}_\mathbf{x}\mathbf{u} = \underbrace{\mathbf{curl}_\mathbf{x}\left(\mathbf{u}_\varphi\right)}_{=0} + \mathbf{curl}_\mathbf{x}\left(\mathbf{u}_\mathbf{\Psi}\right) = \underbrace{\nabla\left(\mathrm{div}_\mathbf{x}\mathbf{\Psi}\right) - \mathbf{\Delta\Psi}}_{+\text{gauge condition}}.$$

The continuous case (1/2)

Hodge-Helmholtz decomposition

Hodge-Helmholtz decomposition

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- Usually computed by potentials extraction
- Preservation of a curl or a divergence
  - Preserving a curl $\approx$ preserving $\boldsymbol{\Psi}$ or $\mathbf{curl}_\mathbf{x}\boldsymbol{\Psi}$
  - Preserving a divergence $\approx$ preserving φ or $\nabla_\mathbf{x}\varphi$.
- Characterization of divergence/curl free vector fields
  - Curl free vectors are such that $\mathbf{u}_{\boldsymbol{\Psi}} = 0$
  - Divergence free vectors are such that $\mathbf{u}_\varphi = 0$
- Uniqueness/orthogonality depends on boundary conditions

The continuous case (2/2)

Structure of the wave system that is preserved

$$\begin{cases} \partial_t p + \operatorname{div}_{\mathbf{x}} \mathbf{u} = 0 \\ \partial_t \mathbf{u} + c^2 \nabla p = 0 \\ + \text{ Boundary conditions } p_b \text{ and } \mathbf{u}_b \end{cases}$$

Structure preserved and long time limit

Given an initial condition $\mathbf{u}_0$, a uniform p_b and $\mathbf{u}_b$ such that $\displaystyle\int_{\partial\Omega} \mathbf{u}_b \cdot \mathbf{n} = 0$, then

- For any $\mathbf{u}$,

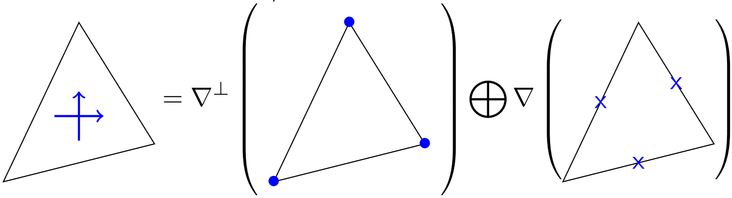
$$\begin{cases} \Delta\varphi = \operatorname{div}_{\mathbf{x}} \mathbf{u} \\ \nabla\varphi \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} - \mathbf{u}_b \cdot \mathbf{n} \end{cases} \tag{1}$$

  has a unique solution up to a constant.
- $\mathbf{u}_{\boldsymbol{\Psi}}(\mathbf{x}, t) := \mathbf{u}(\mathbf{x}, t) - \nabla_{\mathbf{x}} \varphi(\mathbf{x}, t)$ is constant
- the long time limit is $(p_b, \mathbf{u}_{\boldsymbol{\Psi}}(0))$.

Triangular case

- Discrete Hodge-Helmholtz decomposition on triangles

$$\mathbf{d}\mathbb{P}_0/\mathbb{R}^2 = \nabla^{\perp}\mathbb{P}_1 \oplus \nabla\mathbb{CR}$$



- Finite element spaces
  - $\mathbf{d}\mathbb{P}_0$: piecewise constant vectors
  - $\mathbb{P}_1$: continuous finite element space
  - $\mathbb{CR}$: Crouzeix-Raviart finite element space
- Introduced in Arnold, 1989
- Used for low Mach number/long time behaviour of the wave system
  - S. Dellacherie et. al, 2009
  - J. Jung, V.P., 2021

Structure preserved for the wave system

$$\tilde{F} = \begin{pmatrix} \dfrac{\mathbf{u}_L \cdot \mathbf{n} + \mathbf{u}_R \cdot \mathbf{n}}{2} \\ \dfrac{p_L \mathbf{n} + p_R \mathbf{n}}{2} \end{pmatrix} + \frac{c}{2} \begin{pmatrix} 1 & 0 \\ 0 & D(\mathbf{n}) \end{pmatrix} \begin{pmatrix} p_L - p_R \\ \mathbf{u}_L - \mathbf{u}_R \end{pmatrix}$$

- Godunov' scheme : $D(\mathbf{n}) = \mathbf{n}\mathbf{n}^T$
- Lax-Friedrich scheme : $D(\mathbf{n}) = \mathrm{I}_d$

Partial preservation of the Hodge-Helmholtz decomposition

On triangles, with the Godunov' scheme, if

$$\mathbf{u}_0 \in \mathbf{d}\mathbb{P}_0 = \nabla^\perp \psi \oplus \nabla \varphi,$$

with $\psi \in \mathbb{P}_1$ and $\varphi \in \mathbb{CR}$, then $\nabla^\perp \psi$ is preserved.

Structure preserved for the wave system

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Partial preservation of the Hodge-Helmholtz decomposition

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But which curl is preserved ?

Curl preservation

Adjoint curl $\left(\nabla^\perp\right)^\star$

- Definition of the curl

$$\begin{array}{rcccc} \nabla^\perp & : & \mathbb{P}_1 & \longmapsto & \mathbf{d}\mathbb{P}_0 \\ & & \varphi & \longmapsto & \nabla^\perp \varphi \end{array}$$

- Definition of the adjoint discrete curl

$$\forall \mathbf{u} \in \mathbf{d}\mathbb{P}_0, \varphi \in \mathbb{P}_1 \qquad \int_\Omega \varphi \left(\nabla^\perp\right)^\star \mathbf{u} = \int_\Omega \nabla^\perp \varphi \cdot \mathbf{u}$$

- $\left(\nabla^\perp\right)^\star \circ \Pi_{\mathbf{d}\mathbb{P}_0}$ is second order accurate.
- $\left(\nabla^\perp\right)^\star$ is preserved by the Godunov' scheme

Curl preservation

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- Functional analysis : a derivation operator is decreasing the order, and decreasing the regularity

Curl preservation

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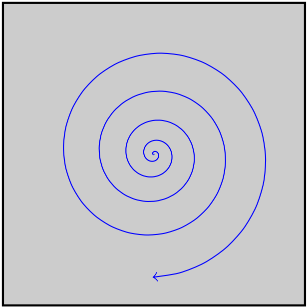
- Functional analysis : a derivation operator is decreasing the order, and decreasing the regularity
- Differential geometry : a derivation operator is an operator that ensures the Leibniz rule

Numerical results

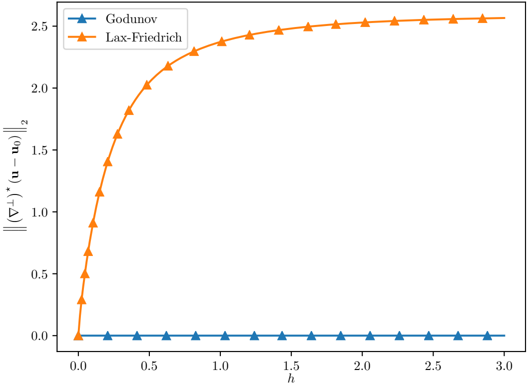
Preservation of the curl (periodic domain)



$$\begin{cases} \rho = \rho_\infty \\ \mathbf{u}_x = -\dfrac{y}{r_0} \mathrm{e}^{-\bar{r}^2} \\ \mathbf{u}_y = \dfrac{x}{r_0} \mathrm{e}^{-\bar{r}^2} \end{cases}$$

Numerical results

Preservation of the curl (periodic domain)



Continuous and discrete conforming de-Rham complex

de-Rham complex

- Starting from

$$\nabla \times (\nabla f) = 0 \qquad \nabla \cdot (\nabla \times \mathbf{u}) = 0$$

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- The operators are written in the smooth de-Rham complex

$$\mathscr{C}^\infty \xrightarrow{\quad\nabla\quad} \mathscr{C}^\infty \otimes \mathbb{R}^3 \xrightarrow{\quad\nabla\times\quad} \mathscr{C}^\infty \otimes \mathbb{R}^3 \xrightarrow{\quad\nabla\cdot\quad} \mathscr{C}^\infty$$

- Two-dimensional version

$$\left\{ \mathscr{C}^\infty \xrightarrow[\nabla^\perp]{\quad\nabla\quad} \mathscr{C}^\infty \otimes \mathbb{R}^2 \xrightarrow[-\nabla\cdot]{\quad\nabla^\perp\cdot\quad} \mathscr{C}^\infty \right.$$

de-Rham complex

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$$\nabla \times (\nabla f) = 0 \qquad \nabla \cdot (\nabla \times \mathbf{u}) = 0$$

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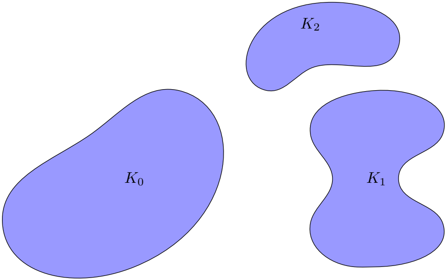
- Natural question :

Is the sequence exact ?

$\ker \nabla$	$\ker \nabla \times / \operatorname{Range} \nabla$
$\ker \nabla \cdot / \operatorname{Range} \nabla\times$	$\operatorname{Range} \nabla\cdot$

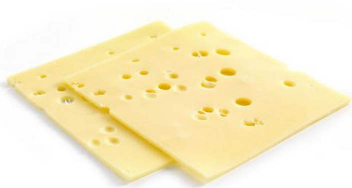
Dimension of $\ker \nabla$

- $\nabla f = 0 \quad \Longrightarrow ?$



$$\dim \ker \nabla = \#\{\text{connected components}\} = b_0$$

Topology of Emmental



$b_1 = \dim\left(\ker \nabla \times / \operatorname{Range} \nabla\right)$
(number of holes in a slice)

$b_2 = \dim\left(\ker\left(\nabla \cdot / \operatorname{Range} \nabla\times\right)\right)$
(number of cavities in the wheel)

Discrete conforming de-Rham complex

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H^{\mathrm{curl}} & \xrightarrow{\nabla\times} & H^{\mathrm{div}} & \xrightarrow{\nabla\cdot} & L^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{P}_{k+1} & \xrightarrow{\nabla_h} & \mathbb{N}_{k+1} & \xrightarrow{\nabla_h\times} & \mathbb{RT}_{k+1} & \xrightarrow{\nabla_h\cdot} & \mathrm{d}\mathbb{P}_k
\end{array}
$$

- $\mathbb{P}_{k+1}$: continuous finite element
- $\mathbb{N}_{k+1}$: tangential conforming (edge Nédélec)
- $\mathbb{RT}_{k+1}$: normal conforming (Raviart-Thomas/face Nédélec)
- $\mathrm{d}\mathbb{P}_k$: discontinuous finite element

Usual properties (Arnold, 2018)

- Approximation property
- Sub-complex property
- bounded cochain projection (commutation between projections and exterior derivative)

Discrete conforming de-Rham complex

$$\begin{array}{ccccccc}
H^1 & \xrightarrow{\quad\nabla\quad} & H^{\mathrm{curl}} & \xrightarrow{\quad\nabla\times\quad} & H^{\mathrm{div}} & \xrightarrow{\quad\nabla\cdot\quad} & L^2 \\
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- $\mathrm{d}\mathbb{P}_k$: discontinuous finite element

Discrete harmonic gap property

$$\left\{\begin{array}{l}
b_0 = \dim\left(\ker \nabla_h\right) \\
b_1 = \dim\left(\ker(\nabla_h\times)/\operatorname{Range}\nabla_h\right) \\
b_2 = \dim\left(\ker(\nabla_h\cdot)/\operatorname{Range}(\nabla_h\times)\right) \\
b_3 = \dim\left(\operatorname{Range}\nabla_h\cdot\right)
\end{array}\right.$$

Nonconforming discrete de-Rham complex and preservation of curl or divergence constraints

Triangular case

- Discrete Hodge-Helmholtz decomposition on triangles

$$\mathbf{d}\mathbb{P}_0/\mathbb{R}^2 = \nabla^{\perp}\mathbb{P}_1 \oplus \nabla\mathbb{C}\mathbb{R}$$

- may be rewritten

$$\mathbb{P}_1 \xrightarrow{\nabla_h^{\perp}} \mathbf{d}\mathbb{P}_0 \xrightarrow{\nabla_{\mathscr{D}'}\cdot} \mathrm{d}\mathbb{P}_0\left(\mathcal{F}\right)$$

Triangular case

- Discrete Hodge-Helmholtz decomposition on triangles

$$\mathbf{d}\mathbb{P}_0/\mathbb{R}^2 = \nabla^\perp \mathbb{P}_1 \oplus \nabla \mathbb{CR}$$

- may be rewritten

$$\mathbb{P}_1 \xrightarrow{\quad \nabla_h^\perp \quad} \mathbf{d}\mathbb{P}_0 \xrightarrow{\quad \nabla_{\mathscr{D}'}\cdot \quad} \mathrm{d}\mathbb{P}_0\left(\mathcal{F}\right)$$

Discrete harmonic gap property for the nonconforming complex

$$\begin{cases} 1 = \dim\left(\ker \nabla_h^\perp\right) \\ 2 = \dim\left(\ker\left(\nabla_{\mathscr{D}'}\cdot\right) / \operatorname{Range}\left(\nabla_h^\perp\right)\right) \\ 1 = \dim\left(\operatorname{Range}\left(\nabla_{\mathscr{D}'}\cdot\right)\right) \end{cases}$$

Triangular case

- Discrete Hodge-Helmholtz decomposition on triangles

$$\mathbf{d}\mathbb{P}_0/\mathbb{R}^2 = \nabla^{\perp}\mathbb{P}_1 \oplus \nabla\mathbb{CR}$$

- may be rewritten and generalized

$$\mathbb{P}_{k+1} \xrightarrow{\nabla_h^{\perp}} \mathbf{d}\mathbb{P}_k \xrightarrow{\nabla_{\mathscr{D}'}\cdot} \mathrm{d}\mathbb{P}_{k-1}\left(\mathcal{C}\right) \times \mathrm{d}\mathbb{P}_k\left(\mathcal{F}\right)$$

Discrete harmonic gap property for the nonconforming complex

$$\left\{\begin{array}{l} 1 = \dim\left(\ker \nabla_h^{\perp}\right) \\ 2 = \dim\left(\ker\left(\nabla_{\mathscr{D}'}\cdot\right) / \operatorname{Range}\left(\nabla_h^{\perp}\right)\right) \\ 1 = \dim\left(\operatorname{Range}\left(\nabla_{\mathscr{D}'}\cdot\right)\right) \end{array}\right.$$

Triangular case

- Discrete Hodge-Helmholtz decomposition on triangles

$$\mathbf{d}\mathbb{P}_0/\mathbb{R}^2 = \nabla^{\perp}\mathbb{P}_1 \oplus \nabla\mathbb{CR}$$

- may be rewritten and generalized

$$\mathbb{P}_{k+1} \xrightarrow{\quad \nabla_h \quad} \mathbf{d}\mathbb{P}_k \xrightarrow{\quad \nabla^{\perp}_{\mathscr{D}'}\cdot \quad} \mathrm{d}\mathbb{P}_{k-1}\left(\mathcal{C}\right) \times \mathrm{d}\mathbb{P}_k\left(\mathcal{F}\right)$$

Discrete harmonic gap property for the nonconforming complex

$$\begin{cases} 1 = \dim\left(\ker \nabla_h\right) \\ 2 = \dim\left(\ker\left(\nabla^{\perp}_{\mathscr{D}'}\cdot\right) / \operatorname{Range}\left(\nabla_h\right)\right) \\ 1 = \dim\left(\operatorname{Range}\left(\nabla^{\perp}_{\mathscr{D}'}\cdot\right)\right) \end{cases}$$

Quadrangular case

Low order

- Triangular non conforming complex

$$\mathbb{P}_{k+1} \xrightarrow[\nabla_h]{\nabla_h^\perp} \mathbf{d}\mathbb{P}_k \xrightarrow[\nabla_{\mathscr{D}'}^\perp \cdot]{\nabla_{\mathscr{D}'} \cdot} \mathrm{d}\mathbb{P}_{k-1}(\mathcal{C}) \times \mathrm{d}\mathbb{P}_k(\mathcal{F})$$

- Quadrangular non conforming complex

$$\mathbb{Q}_1 \xrightarrow{\nabla_h^\perp} \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ -y \end{pmatrix}\right) \xrightarrow{\nabla_{\mathscr{D}'} \cdot} \mathrm{d}\mathbb{P}_k(\mathcal{F})$$

- On the two-dimensional torus

Discrete harmonic gap property for the nonconforming complex

$$\begin{cases} 1 = \dim\left(\ker \nabla_h\right) \\ 2 = \dim\left(\ker\left(\nabla_{\mathscr{D}'}^\perp \cdot\right) / \mathrm{Range}\left(\nabla_h\right)\right) \\ 1 = \dim\left(\mathrm{Range}\left(\nabla_{\mathscr{D}'}^\perp \cdot\right)\right) \end{cases}$$

Quadrangular case

- Triangular non conforming complex

$$\mathbb{P}_{k+1} \xrightarrow[\nabla_h]{\nabla_h^\perp} \mathbf{d}\mathbb{P}_k \xrightarrow[\nabla_{\mathscr{D}'}^\perp\cdot]{\nabla_{\mathscr{D}'}\cdot} \mathrm{d}\mathbb{P}_{k-1}(\mathcal{C}) \times \mathrm{d}\mathbb{P}_k(\mathcal{F})$$

- Quadrangular non conforming complex

$$\left\{\begin{array}{l} \mathbb{Q}_{k+1} \xrightarrow{\nabla_h^\perp} \mathbf{d}\mathbb{B}_{k+1}^{\mathrm{div}} \xrightarrow{\nabla_{\mathscr{D}'}\cdot} \mathrm{d}\check{\mathbb{Q}}_k(\mathcal{C}) \times \mathrm{d}\mathbb{P}_k(\mathcal{F}) \\ \mathbb{Q}_{k+1} \xrightarrow{\nabla_h} \mathbf{d}\mathbb{B}_{k+1}^{\mathrm{curl}} \xrightarrow{\nabla_{\mathscr{D}'}^\perp\cdot} \mathrm{d}\check{\mathbb{Q}}_k(\mathcal{C}) \times \mathrm{d}\mathbb{P}_k(\mathcal{F}) \end{array}\right.$$

- On the two-dimensional torus

Discrete harmonic gap property for the nonconforming complex

$$\left\{\begin{array}{l} 1 = \dim\left(\ker \nabla_h\right) \\ 2 = \dim\left(\ker\left(\nabla_{\mathscr{D}'}^\perp\cdot\right) / \operatorname{Range}\left(\nabla_h\right)\right) \\ 1 = \dim\left(\operatorname{Range}\left(\nabla_{\mathscr{D}'}^\perp\cdot\right)\right) \end{array}\right.$$

Conservation of an curl

Discrete conservation of a curl

Consider the discontinuous Galerkin discretization

$$\text{Find } \mathbf{u} \in \mathbf{d}\mathbb{B}_k \quad \forall \mathbf{v} \in \mathbf{d}\mathbb{B}_k$$
$$\sum_{C\in\mathcal{C}} \int_C \mathbf{v}\cdot\partial_t \mathbf{u} - \sum_{C\in\mathcal{C}} \int_C \overline{\mathbf{G}} : \nabla \mathbf{v} + \sum_{S\in\mathcal{S}} \int_S [\![\mathbf{v}]\!] \cdot \widetilde{\mathbf{G}} = 0$$

if

- $\overline{\mathbf{G}} = g\mathrm{I}_d$
- $\mathbf{d}\mathbb{B} = \mathbf{d}\mathbb{B}_k^{\mathrm{div}}$
- the numerical flux is

$$\widetilde{\mathbf{G}} = \frac{g_L\mathbf{n} + g_R\mathbf{n}}{2} + \frac{\lambda}{2}\,\mathbf{n}\mathbf{n}^T\,(\mathbf{u}_L - \mathbf{u}_R)$$

then $\left(\nabla^\perp\right)^\star \mathbf{u}$ is preserved

Conservation of an divergence

Discrete conservation of a divergence

Consider the discontinuous Galerkin discretization

$$\text{Find } \mathbf{u} \in \mathbf{d}\mathbb{B}_k \quad \forall \mathbf{v} \in \mathbf{d}\mathbb{B}_k$$
$$\sum_{C\in\mathcal{C}}\int_C \mathbf{v}\cdot\partial_t\mathbf{u} - \sum_{C\in\mathcal{C}}\int_C \overline{\mathbf{G}} : \nabla\mathbf{v} + \sum_{S\in\mathcal{S}}\int_S [\![\mathbf{v}]\!]\cdot\widetilde{\mathbf{G}} = 0$$

If

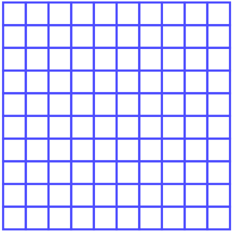
- $\overline{\mathbf{G}} = \mathbf{g}\times$
- $\mathbf{d}\mathbb{B} = \mathbf{d}\mathbb{B}_k^{\mathrm{curl}}$
- the numerical flux is

$$\widetilde{\mathbf{G}} = \frac{g_L\mathbf{n}^\perp + g_R\mathbf{n}^\perp}{2} + \frac{\lambda}{2}\left(\mathrm{I} - \mathbf{n}\mathbf{n}^T\right)(\mathbf{u}_L - \mathbf{u}_R)$$

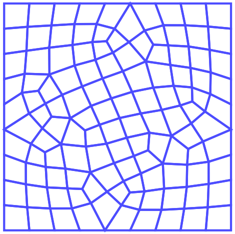
then $\nabla^\star$ is preserved.

Numerical results

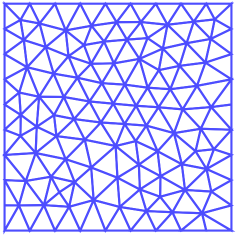
Meshes considered



Cartesian



Unstructured quad

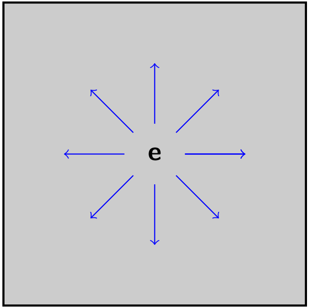


Unstructured triangles

Two dimensional Maxwell system

Preservation of the divergence

$$\begin{cases} \partial_t b + \nabla^\perp \cdot \mathbf{e} = 0 \\ \partial_t \mathbf{e} + \nabla^\perp b = 0 \end{cases}$$

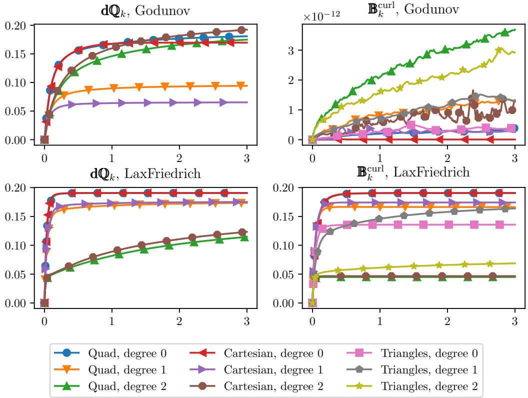


$$\begin{cases} b(\mathbf{x}) = 0 \\ \mathbf{e}_x(\mathbf{x}) = \overline{x}\, \mathrm{e}^{-\overline{r}^2/2} \\ \mathbf{e}_y(\mathbf{x}) = \overline{y}\, \mathrm{e}^{-\overline{r}^2/2}, \end{cases}$$

Two dimensional Maxwell system

Preservation of the divergence

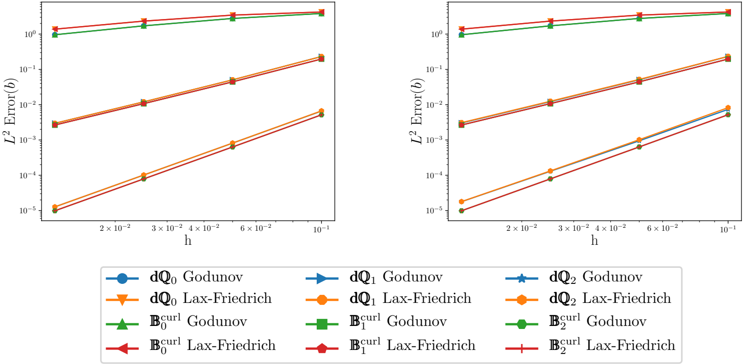
$$\begin{cases} \partial_t b + \nabla^\perp \cdot \mathbf{e} = 0 \\ \partial_t \mathbf{e} + \nabla^\perp b = 0 \end{cases}$$



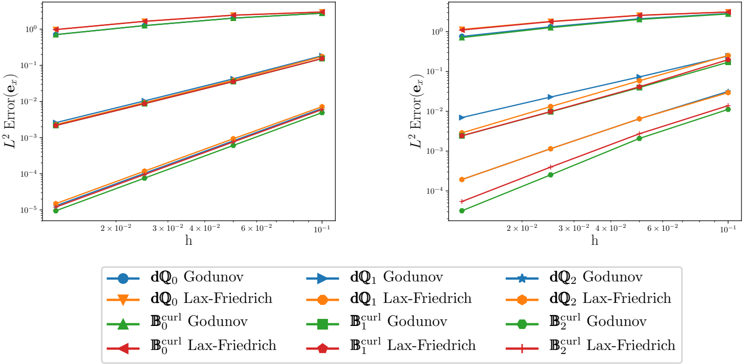
Two dimensional Maxwell system

Convergence

$$
\left\{
\begin{aligned}
b(\mathbf{x},\, t) &= \frac{\omega}{c^2}\cos\left(k_\perp\,\pi\,y\right)\,\sin\left(k_{||}\,\pi\,x - \omega\,t\right) \\
\mathbf{e}_x(\mathbf{x},\, t) &= -k_\perp\,\pi\,\sin\left(k_\perp\,\pi\,y\right)\,\cos\left(k_{||}\,\pi\,x - \omega\,t\right) \qquad {\color{blue} +2K_0\,\alpha\,\bar{x}\,\frac{\mathrm{e}^{-\alpha/(1-\bar{r}^2)}}{(1-\bar{r}^2)^2}} \\
\mathbf{e}_y(\mathbf{x},\, t) &= k_{||}\,\pi\,\cos\left(k_\perp\,\pi\,y\right)\,\sin\left(k_{||}\,\pi\,x - \omega\,t\right) \qquad {\color{blue} +2K_0\,\alpha\,\bar{y}\,\frac{\mathrm{e}^{-\alpha/(1-\bar{r}^2)}}{(1-\bar{r}^2)^2}}
\end{aligned}
\right.
$$



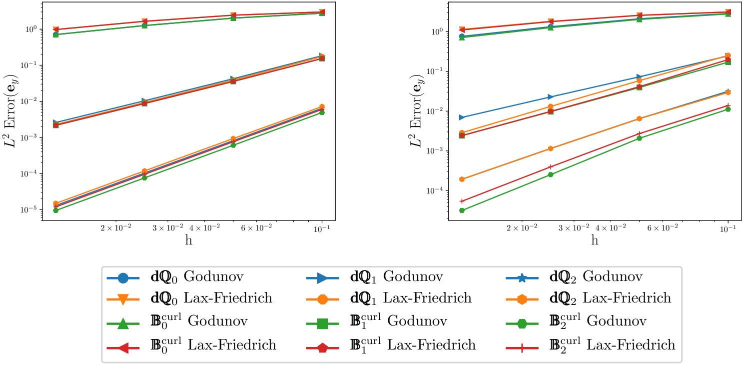
$$
\begin{cases}
b(\mathbf{x},\, t) = \dfrac{\omega}{c^2} \cos\left(k_\perp \pi y\right) \sin\left(k_{||} \pi x - \omega t\right) \\
\mathbf{e}_x(\mathbf{x},\, t) = -k_\perp \pi \sin\left(k_\perp \pi y\right) \cos\left(k_{||} \pi x - \omega t\right) \qquad +2K_0\, \alpha\, \bar{x}\, \dfrac{\mathrm{e}^{-\alpha/(1-\bar{r}^2)}}{(1-\bar{r}^2)^2} \\
\mathbf{e}_y(\mathbf{x},\, t) = k_{||} \pi \cos\left(k_\perp \pi y\right) \sin\left(k_{||} \pi x - \omega t\right) \qquad +2K_0\, \alpha\, \bar{y}\, \dfrac{\mathrm{e}^{-\alpha/(1-\bar{r}^2)}}{(1-\bar{r}^2)^2}
\end{cases}
$$



Two dimensional Maxwell system

Convergence

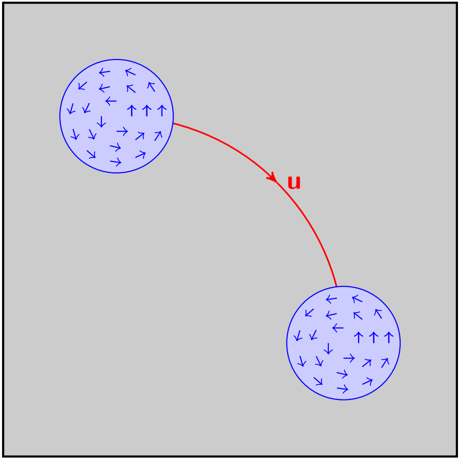
$$
\begin{cases}
b(\mathbf{x},t) = \dfrac{\omega}{c^2}\cos\left(k_\perp \pi y\right)\sin\left(k_{||}\pi x - \omega t\right) \\
\mathbf{e}_x(\mathbf{x},t) = -k_\perp \pi \sin\left(k_\perp \pi y\right)\cos\left(k_{||}\pi x - \omega t\right) \quad {\color{blue} +2K_0\,\alpha\,\bar{x}\,\dfrac{\mathrm{e}^{-\alpha/(1-\bar{r}^2)}}{(1-\bar{r}^2)^2}} \\
\mathbf{e}_y(\mathbf{x},t) = k_{||}\pi \cos\left(k_\perp \pi y\right)\sin\left(k_{||}\pi x - \omega t\right) \quad {\color{blue} +2K_0\,\alpha\,\bar{y}\,\dfrac{\mathrm{e}^{-\alpha/(1-\bar{r}^2)}}{(1-\bar{r}^2)^2}}
\end{cases}
$$



Two dimensional Induction equation

Regular rotating magnetic loop

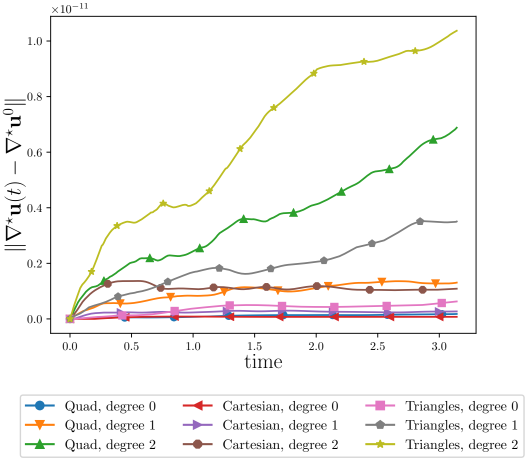
$$\partial_t \mathbf{b} + \nabla^{\perp}\left(\det(\mathbf{u},\mathbf{b})\right) + \mathbf{u}\nabla\cdot\mathbf{b} = 0$$



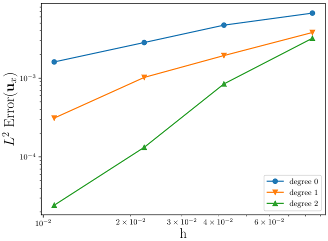
$$\mathbf{b}^0(\mathbf{x}) = \begin{cases} \text{if } \bar{r} < r_0 & \begin{cases} \mathbf{u}_x = -2K_0\,\alpha\bar{y}\,\dfrac{\mathrm{e}^{-\alpha/(1-\bar{r}^2)}}{(1-\bar{r})^2} \\ \mathbf{u}_y = \quad 2K_0\,\alpha\bar{x}\,\dfrac{\mathrm{e}^{-\alpha/(1-\bar{r}^2)}}{(1-\bar{r})^2} \end{cases} \\ \text{if } \bar{r} \geq r_0 & 0, \end{cases}$$

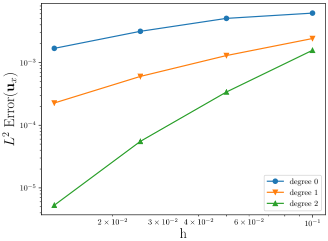
Two dimensional Induction equation

Divergence preservation



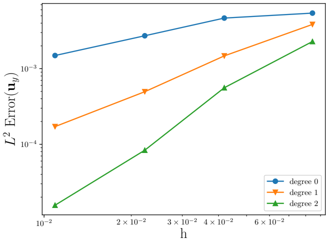
Two dimensional Induction system
Convergence

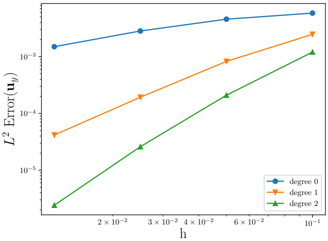




Two dimensional Induction system

Convergence





Conclusion

Conclusion

- On Triangles
  - The space $\mathbf{d}\mathbb{P}_k$ can be naturally included in a discrete distributional de-Rham complex.
  - With mild assumption on the numerical flux, divergence or curl constraints are preserved with the discontinuous Galerkin method.
- On Quads
  - Discrete spaces can be also built for quads.
  - Same assumptions as for triangles on the numerical flux ensure the conservation of divergence or curl.
- Main ideas
  - Some of the discrete operators are taken in the distribution sense.
  - Curl and divergence that are preserved are defined in the adjoint sense.
  - Galerkin methods are naturally well suited with adjoint differential operators.
- Prospects
  - Applications to other systems (MHD, low Mach number flows).
  - Higher order involutions (elasticity, relativity).
  - Extensions to 3d.