Wave and spectral solvers with self-gravitation for radially symmetric adiabatic backgrounds in helioseismology

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2. Equations and general question
3. PART I : Wave solver - Resolution in radial symmetry
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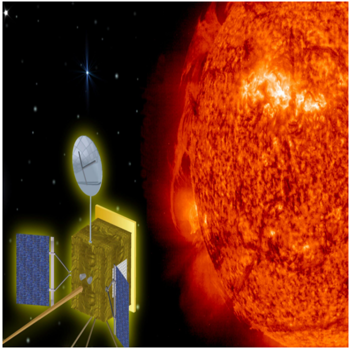
Introduction to helioseismology

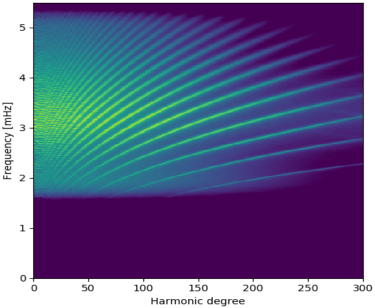


SDO/HMI. Credit: NASA/SDO

Helioseismology reconstructs the subsurface structures and dynamics of the Sun from oscillations observed in the visible layer of the photosphere.

Helioseismic observables

Filtered **HMI power spectrum** of Doppler velocity $P_\ell(\omega)$ showing standing acoustic oscillations



Credit: SDO/NASA

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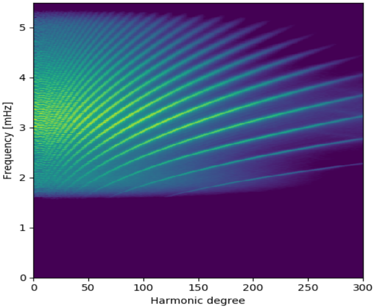

Credit: SDO/NASA

Synthetic observables

$$\mathbb{E}[P_\ell(\omega)] \sim \operatorname{Im} G \quad \textit{(convenient source assumption.)}$$

G is Green kernel to a wave equation

$$\left(-\sigma^2 - \mathcal{L}\right) G = \delta\Big(\mathbf{x} - \mathbf{y}\Big).$$

General objectives

Taking into account effect of **the perturbation of gravity** on acoustic oscillations :

- Compute **G** with $\mathcal{L}$ using accurate and robust numerical methods.
- Create an eigensolver to model the eigenvalues.

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Time-harmonic solar wave equation in Eulerian-Lagragian description

Denote by $\left\{ \begin{array}{ll} \xi & = \text{small Lagrangian displacement} \\ \delta_\phi & = \text{perturbation to gravitational potential} \end{array} \right\}$ on top of a stationary self-gravitating adiabatic background without flow

Full equation

(ξ, δ_ϕ) satisfies the time-harmonic Galbrun's equation without rotation and flow.

$$\left\{ \begin{array}{l} -\rho_0\,\sigma^2\xi - \mathcal{L}\xi + \rho_0\nabla\delta_\phi = \mathbf{F}, \\ \Delta\delta_\phi = -4\pi G\,\nabla\cdot(\rho_0\xi)\,. \end{array} \right.$$

$$\mathcal{L} = \nabla(\gamma\, p_0 \nabla_{\mathbf{x}}\cdot\xi) - (\nabla p_0)(\nabla_{\mathbf{x}}\cdot\xi) + \nabla[(\xi\cdot\nabla)p_0] - (\xi\cdot\nabla)\nabla p_0 - \rho_0\,(\xi\cdot\nabla_{\mathbf{x}})\,\nabla\phi_0.$$

The interior of the Sun is characterized by :

- ✓ pressure p_0,
- ✓ density ρ_0,
- ✓ adiabatic index γ
- ✓ gravitational potential ϕ_0 satisfying

$$\Delta\phi_0 = 4\pi G\rho_0\,, \quad \phi_0 \to 0,\ |\mathbf{x}| \to \infty\,.$$

Lynden-Bell, D., & Ostriker, J. P. (1967).
On the stability of differentially rotating bodies
Monthly Notices of the Royal Astronomical Society, 136(3)

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Cowling approximation

Ignoring perturbation to gravitational potential δ_ϕ,

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What is the effect of Cowling approximation on Green's kernel and position of eigenvalues ?

Time-harmonic solar wave equation in Eulerian-Lagragian description

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Cowling approximation

Ignoring perturbation to gravitational potential δ_ϕ,

$$-\rho_0\,\sigma^2\xi \;-\; \mathcal{L}\xi \;=\; \mathbf{F}\,.$$

Background parameters in Standard Solar Models are radially symmetric.

What is the effect of Cowling approximation on Green's kernel and position of eigenvalues ?

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Boundary value problems (BVP)

We denote $\mathbb{B}_S := \{|\mathbf{x}| \leq R_S\}$, R_S = height at the end of solar model S.

BVP 1

$$
\begin{cases}
-(\sigma^2 + \mathcal{L})\xi + \nabla\delta_\phi = \dfrac{\mathbf{F}}{\rho_0}, \quad \text{on } \mathbb{B}_S\,, \\
\Delta\delta_\phi = \begin{cases} -4\pi G\,\nabla\cdot(\rho_0\xi)\,, & \text{on } \mathbb{B}_S \\ 0\,, & \text{on } \mathbb{R}^3 \setminus \mathbb{B}_S \end{cases}, \\
\xi\cdot\boldsymbol{n} = 0\,, \quad \text{at } \mathbf{x} \in \partial\mathbb{B}_S, \qquad (\star) \\
[\delta_\phi] = [\partial_n\delta_\phi] = 0, \quad \text{as } \mathbf{x} \in \partial\mathbb{B}_S, \\
\delta_\phi \to 0\,, \quad \text{as } \mathbf{x} \to \infty\,.
\end{cases}
$$

BVP 2

$$
\begin{cases}
-(\sigma^2 + \mathcal{L})\xi + \nabla\delta_\phi = \dfrac{\mathbf{F}}{\rho_0}, \quad \text{on } \mathbb{B}_S\,, \\
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\delta_p = 0\,, \quad \text{at } \mathbf{x} \in \partial\mathbb{B}_S, \qquad (\star\star) \\
[\delta_\phi] = 0, \quad [\partial_n\delta_\phi] = -4\,\pi\,G\,\rho_0\,\xi\cdot\boldsymbol{n}, \quad \text{as } \mathbf{x} \in \partial\mathbb{B}_S, \\
\delta_\phi \to 0\,, \quad \text{as } \mathbf{x} \to \infty\,.
\end{cases}
$$

BC (⋆) and (⋆⋆)

- employ for eigenvalue investigation, cf. Gyre , ADIPLS.
- below cut-off frequency (5.3 mHz).

δ_p : the perturbation to pressure

$$\boxed{\delta_p = -\xi\cdot\nabla p_0 - \rho_0 c_0^2 \nabla\cdot\xi.}$$

Boundary value problems

BVP 1

$$\begin{cases} -(\sigma^2 + \mathcal{L})\xi + \nabla\delta_\phi = \dfrac{\mathbf{F}}{\rho_0}, & \text{on } \mathbb{B}_S\,, \\ \Delta\delta_\phi = \begin{cases} -4\pi G\,\nabla\cdot(\rho_0\xi)\,, & \text{on } \mathbb{B}_S \\ 0\,, & \text{on } \mathbb{R}^3 \setminus \mathbb{B}_S \end{cases}, & \\ \xi\cdot\boldsymbol{n} = 0\,, & \text{at } \mathbf{x}\in\partial\mathbb{B}_S, \\ [\delta_\phi] = [\partial_n\delta_\phi] = 0, & \text{as } \mathbf{x}\in\partial\mathbb{B}_S, \\ \delta_\phi \to 0\,, & \text{as } \mathbf{x}\to\infty\,. \end{cases}$$

BVP 2

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- Well-posedness of BVP 1 is investigated in,

Halla, M., & Hohage, T. (2021). *On the well-posedness of the damped time-harmonic Galbrun equation and the equations of stellar oscillations* SIAM Journal on Mathematical Analysis, 53(4), 4068-4095

- $(*)$ & $(**)$ also employed to describe oscillations in a self-gravitating Earth,

Chaljub, E., & Valette, B. (2004). *Spectral element modelling of three-dimensional wave propagation in a self-gravitating Earth with an arbitrarily stratified outer core.* Geophysical Journal International.

Gharti, H., & Eaton, W., & Tromp, J. (2023). *Spectral-infinite-element simulations of seismic wave propagation in self-gravitating, rotating 3D Earth models.* Geophysical Journal International.

Resolution in radially symmetric background

Expansion of unknowns in Vector spherical harmonics

$$\xi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m(r)\, \mathbf{P}_\ell^m(\hat{\mathbf{x}}) + \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} b_\ell^m(r)\, \mathbf{B}_\ell^m(\hat{\mathbf{x}}) + c_\ell^m(r)\, \mathbf{C}_\ell^m(\hat{\mathbf{x}}), \quad \Big| \quad \delta_\phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} d_\ell(r)\, \mathrm{Y}_\ell^m(\hat{\mathbf{x}}).$$

Similar for **F** with (f_ℓ, g_ℓ, h_ℓ).

Vector spherical harmonics (VSH)

An **orthonormal basis** for $L^2(\mathbb{R}^3)^3$ vectors defined in terms of the ***scalar spherical harmonics*** Y_ℓ^m and *tangential gradient* $\nabla_{\mathbb{S}^2}$,

$$\mathbf{P}_\ell^m(\widehat{\mathbf{x}}) = \mathrm{Y}_\ell^m(\widehat{\mathbf{x}})\, \mathbf{e}_r, \quad \ell = 0, 1, \ldots;$$

$$\mathbf{B}_\ell^m(\widehat{\mathbf{x}}) = \frac{\nabla_{\mathbb{S}^2} \mathrm{Y}_\ell^m}{\sqrt{\ell(\ell+1)}}, \qquad \mathbf{C}_\ell^m(\widehat{\mathbf{x}}) = -\frac{\mathbf{e}_r \times \nabla_{\mathbb{S}^2} \mathrm{Y}_\ell^m}{\sqrt{\ell(\ell+1)}}, \qquad \ell = 1, 2, \ldots.$$

Resolution in radial symmetry - Modal equations

Complete model

$$\begin{cases} -(\sigma^2 \rho_0 + \mathcal{L})\,\boldsymbol{\xi} \ + \ \nabla \delta_\phi = \mathbf{F}, \\ \Delta \delta_\phi = -4\pi G\, \nabla \cdot (\rho_0 \boldsymbol{\xi})\,. \end{cases}$$

$\mathfrak{f}_\ell$ depends on f_ℓ, g_ℓ, $\mathfrak{g}_\ell$ is a multiple of g_ℓ.

Coefficients of unknowns and RHS

$$\boldsymbol{\xi} \leftrightarrow (a_\ell^m, b_\ell^m, c_\ell^m), \qquad \delta_\phi \leftrightarrow d_\ell^m$$

$$\mathbf{F} \leftrightarrow (f_\ell^m, g_\ell^m, h_\ell^m)$$

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$$(a_\ell, d_\ell)\;,\;(f_\ell, g_\ell) \underset{\text{determines}}{\mapsto} b_\ell,$$

$$h_\ell \underset{\text{multiple of}}{\mapsto} c_\ell$$

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$$\Rightarrow \begin{cases} \left(\hat{q}_\ell \, \partial_r^2 + q_\ell \, \partial_r + \tilde{q}_\ell\right) a_\ell + \left(Q \, \partial_r + \tilde{Q}\right) d_\ell = \mathfrak{f}_\ell \, , \\ \left(r^2 \partial_r^2 + 2r\partial_r + \tilde{m}_\ell\right) d_\ell + \left(P \, r \, \partial_r + \tilde{P}\right) a_\ell = \mathfrak{g}_\ell \, . \end{cases}$$

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h_ℓ $\underset{\text{multiple of}}{\mapsto}$ c_ℓ

Coefficients of unknowns and RHS

$\xi \leftrightarrow (a_\ell^m, b_\ell^m, c_\ell^m)$, $\delta_\phi \leftrightarrow d_\ell^m$

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$\mathfrak{f}_\ell$ depends on f_ℓ, g_ℓ, $\mathfrak{g}_\ell$ is a multiple of g_ℓ.

NB : system ***rational in*** σ^2.

Resolution in radial symmetry - Modal equations

Complete model

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$\mathfrak{f}_\ell$ depends on f_ℓ, g_ℓ, $\quad \mathfrak{g}_\ell$ is a multiple of g_ℓ.

Cowling approximation

$$-(\sigma^2\rho_0 + \mathcal{L})\,\boldsymbol{\xi} = \mathbf{F}$$

$$\Longrightarrow \quad \left(\hat{q}_\ell\,\partial_r^2 + q_\ell\,\partial_r + \tilde{q}_\ell\right) a_\ell = \mathfrak{f}_\ell\,.$$

With attenuation ($\operatorname{Im}\sigma > 0$)

- coefficients are **continuous** on $r > 0$,
- the ODEs have regular singularities at $r = 0$.

Singular regular boundary conditions at $r = 0$

- **Equations in 3D** have **no singularity** at the origin $\mathbf{x} = 0$.
- Due to the singularity of the spherical coordinates, the **modal equations** have **regular singularities** at $r = 0$.

Regular indicial boundary condition at $r = 0$ chooses non-singular solution :

$$\begin{cases} r a_\ell' = 0\,, \\ r d_\ell' - \ell d_\ell = 0 \end{cases} \qquad \text{at} \qquad r = 0.$$

Barucq, H., Faucher, F., Fournier, D., Gizon, L., & Pham, H. (2021).
Outgoing modal solutions for Galbrun's equation in helioseismology
Journal of Differential Equation

Unno, W., Osaki, Y., Ando, H., & Shibahashi, H. (1979)).
Nonradial oscillations of stars.
Tokyo: University of Tokyo Press.

Boundary conditions for r_b and $r_{\max}$

Boundary condition for a_ℓ :

We employ a free-surface or a Dirichlet condition at $r = r_b$ for a_ℓ :

$$r a_\ell' + \left(-2 + \frac{\alpha_{p_0}}{\gamma} r\right) a_\ell = 0 \iff \delta_p = 0 \text{ (free-surface)} \qquad \text{or} \qquad a_\ell = 0 \iff \xi \cdot \mathbf{n} = 0 \text{ (Dirichlet)} \,.$$

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Boundary condition for d_ℓ :

We employ a DtN condition at $r = r_{\max}$ for d_ℓ :

$$r d_\ell' \;+\; (\ell + 1) d_\ell = 0 \qquad \text{on } r = r_{\max} > r_b.$$

Question under consideration

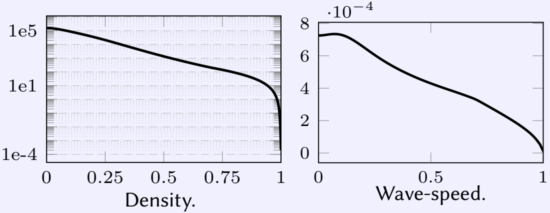
What is the effect of Cowling approximation on Green's kernel and position of eigenvalues ?

Computational framewok

- Solve our problem using HDG code and CG code.
- Implement correct boundary conditions specifically for δ_ϕ to truncate the Poisson equation.
- Numerical implementations in Hawen software (https://ffaucher.gitlab.io/hawen-website/).

Technical problem : Model S for solar parameters.

Both density and wave-speed exponentially decrease near surface layer.



Working equations

Normalized problem

$$\left(r^2\partial_r^2 + (\frac{q_\ell}{\hat{q}_\ell}+1)r\partial_r + \frac{\tilde{q}_\ell}{\hat{q}_\ell}\right) a_\ell + \left(\frac{Q}{\hat{q}_\ell} r\partial_r + \frac{\tilde{Q}}{\hat{q}_\ell}\right) d_\ell = \frac{\delta(r-s)}{\hat{q}_\ell} \text{ on } [0, r_b]; \qquad \text{and}$$

$$\begin{cases} \left(r^2\partial_r^2 + (\frac{m_\ell}{\hat{m}_\ell}+1)r\partial_r + \frac{\tilde{m}_\ell}{\hat{m}_\ell}\right) d_\ell + \left(\frac{P}{\hat{m}_\ell} r\,\partial_r + \frac{\tilde{P}}{\hat{m}_\ell}\right) a_\ell = 0 \text{ on } [0, r_b] \\ \left(r^2\partial_r^2 + (\frac{m_\ell}{\hat{m}_\ell}+1)r\partial_r + \frac{\tilde{m}_\ell}{\hat{m}_\ell}\right) d_\ell = 0 \qquad \text{on } [r_b, r_{\max}] \end{cases}$$

with BC

- at $r = 0$

regular singular BC $\begin{cases} r a_\ell' = 0; \\ r d_\ell' - \ell d_\ell = 0. \end{cases}$

- at $r_b = 1.001$

$a_\ell = 0$ or $r a_\ell' + \left(2 - \frac{\alpha_{p0}}{\gamma} r\right) a_\ell = 0$ **zero-surface pressure**

& Adapted Jump condition

- at $r_{\max}$

$r d_\ell' + (\ell+1) d_\ell = 0$

exact DtN.

A few words about the HDG method

Discretization with Hybrizable Discontinuous Galerkin method

- Based on two different problems: a **global** and a **local** one.
- **Static condensation** for first-order problems without increasing the number of unknowns: the **unknowns** of the global matrix are only the **numerical traces** λ_a and λ_d
- Adapted to complex geometry (**p-adaptivity**)
- Resolution of a **reduced system**

Illustration of the degrees of freedom in one dimensions for polynomial order 3 - From left to righ : CG, DG and HDG method.

Construction of the HDG method

- **1st order formulation** : unknowns $U_h = (a_h, \underbrace{ra_h'}_{v_h}, d_h, \underbrace{rd_h'}_{w_h})^T$ and numerical traces $\Lambda = (\lambda_a, \lambda_d)^T$

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Discretized HDG formulation :

Find $\left((\mathsf{U}^e)_{1\leq e\leq|\mathcal{T}_h|}, \Lambda\right)$ that solve :

$$\begin{cases} \mathbb{B}^e\,\mathsf{U}^e + \mathbb{C}^e\mathcal{R}_e\Lambda = \mathbb{F}^e, & \textbf{Local problem on each cell} \\ \displaystyle\sum_e^{|\mathcal{T}_h|} \mathcal{R}_e^T\left(\mathbb{D}^e\,\mathsf{U}^e + \mathsf{L}^e\mathcal{R}_e\Lambda\right) = 0, & \textbf{Relation for numerical traces} \end{cases}$$

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- **Solve** the **large linear system** $\sum_e \mathcal{R}_e^T (\mathbb{L}_e - \mathbb{D}_e \mathbb{B}_e^{-1} \mathbb{C}_e) \mathcal{R}_e \Lambda = - \sum_e \mathcal{R}_e^T \mathbb{B}_e \mathbb{D}_e^{-1} \mathbb{F}_e.$

Construction of the HDG method

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Find $\left((\mathrm{U}^e)_{1\leq e\leq |\mathcal{T}_h|}, \Lambda\right)$ that solve :

$$\begin{cases} \mathbb{B}^e\, \mathrm{U}^e + \mathbb{C}^e \mathcal{R}_e \Lambda = \mathbb{F}^e, & \textbf{Local problem on each cell} \\ \displaystyle\sum_e^{|\mathcal{T}_h|} \mathcal{R}_e^T \left(\mathbb{D}^e\, \mathrm{U}^e + \mathrm{L}^e \mathcal{R}_e \Lambda\right) = 0, & \textbf{Relation for numerical traces} \end{cases}$$

- **Solve** the **large linear system** $\displaystyle\sum_e \mathcal{R}_e^T (\mathbb{L}_e - \mathbb{D}_e \mathbb{B}_e^{-1} \mathbb{C}_e) \mathcal{R}_e \Lambda = -\sum_e \mathcal{R}_e^T \mathbb{B}_e \mathbb{D}_e^{-1} \mathbb{F}_e.$
- **Solve** the **local linear sytem** to obtain the **volumic solutions** (a_h, d_h)

Construction of the HDG method

- **1st order formulation** : unknowns $U_h = (a_h, \underbrace{r a_h'}_{v_h}, d_h, \underbrace{r d_h'}_{w_h})^T$ and numerical traces $\Lambda = (\lambda_a, \lambda_d)^T$

Discretized HDG formulation :

Find $\left((\mathrm{U}^e)_{1 \le e \le |\mathcal{T}_h|}, \Lambda \right)$ that solve :

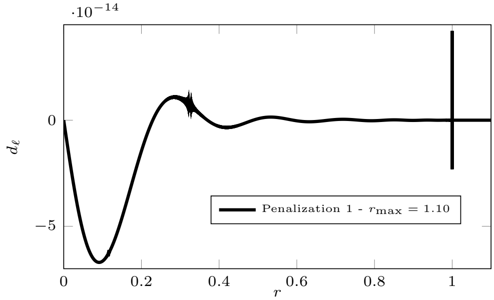
$$\begin{cases} \mathbb{B}^e\, \mathrm{U}^e + \mathbb{C}^e \mathcal{R}_e \Lambda = \mathbb{F}^e, & \textbf{Local problem on each cell} \\ \displaystyle\sum_e^{|\mathcal{T}_h|} \mathcal{R}_e^T \left(\mathbb{D}^e\, \mathrm{U}^e + \mathrm{L}^e \mathcal{R}_e \Lambda \right) = 0, & \textbf{Relation for numerical traces} \end{cases}$$

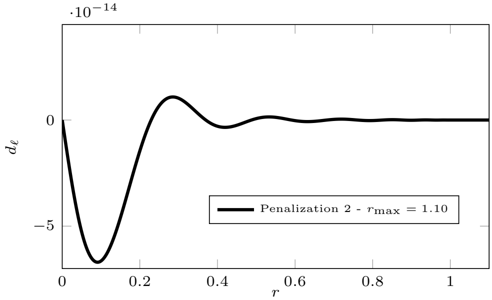
⋆ Essence of the HDG method : the formulation of the numerical fluxes

$$\widehat{v_h}^{(e)} = v_h^{(e)} + \tau_a (a_h^{(e)} - \lambda_a^{(e)})\, \boldsymbol{n}_{\mathfrak{f}}^{(e)} \quad \text{and} \quad \widehat{w_h}^{(e)} = w_h^{(e)} + \tau_d (d_h^{(e)} - \lambda_d^{(e)})\, \boldsymbol{n}_{\mathfrak{f}}^{(e)}, \qquad (1)$$

Result 1 : HDG method and choice of the stabilization parameter

Comparisons of d_ℓ with two different choices of penalization with $r_b = 1.001$ and $r_{\max} = 1.100$. Mode $\ell = 1$ - Frequency 2 Mhz.





Choice 1 : $\tau_a = \tau_d = 1.0$

Choice 2 : $\tau_a = -r\left(\mathfrak{J}'_\ell/\mathfrak{J}_\ell - \mathrm{i}\sqrt{-V_\ell}\,\boldsymbol{n}\right)$;

$$\tau_d = (1 - \boldsymbol{n}\, r)\,, \qquad (\ell = 0)$$
$$\tau_d = 1 - \sqrt{\ell(\ell+1)}\,\boldsymbol{n}\,, \qquad (\ell > 0).$$

Remark : Choice 1 $\tau_a = 1.0$ is adapted for the case with Cowling's approximation.

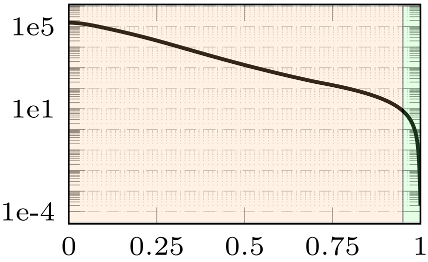
Result 2 : Comparison HDG and CG method

Parameters : $r_{\mathbf{min}} = 0$, $r_b = 1.001$ and $r_{\max} = 1.100$.

Comparisons are carried out between the **two meshes** :

1. Mesh 1 : 2 step sizes, $h_1 = 2e - 04$ and $h_2 = 1e - 04$

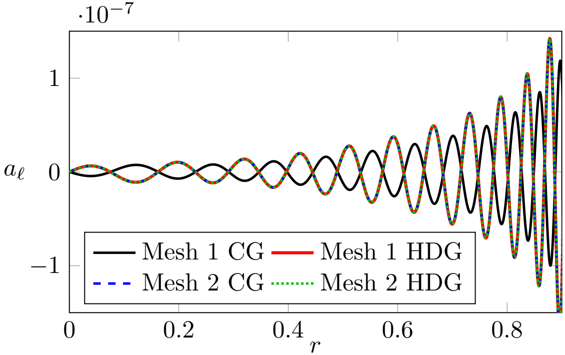
2. Mesh 2 : starts from Mesh 1 with refinement around r_b, $h_3 = 1e - 07$



Density.

Numerical result 2: Comparison HDG vs CG method

		HDG	CG	HDG + CG
BVP 1	$[\delta_\phi] = [\partial_r \delta_\phi] = 0 (= 4\pi G \rho_0 \boldsymbol{\xi} \cdot \boldsymbol{n}), a_\ell = 0.$	Agreement	Agreement	$M1_{HDG} = M2_{HDG} = M1_{CG} = M2_{CG}$
BVP 2	$[\delta_\phi] = 0;\ [\partial_r \delta_\phi] = 4\pi G \rho_0\, \boldsymbol{\xi} \cdot \boldsymbol{n},\ e_\ell = 0.$			

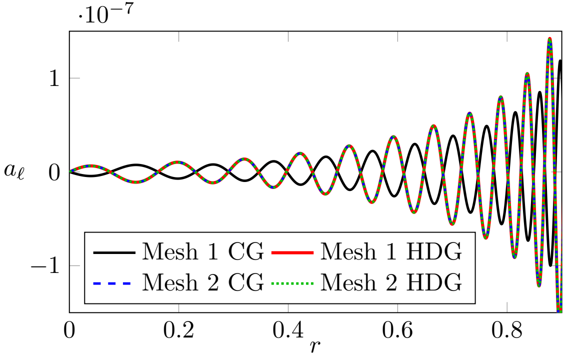


Solution $a_\ell(r, s)$ with Dirac source at $s = 1$ and $\ell = 2$, $\omega/2\pi = 5$ mHz.

Numerical result 2: Comparison HDG vs CG method

		HDG	CG	HDG + CG
BVP 1	$[\delta_\phi] = [\partial_r \delta_\phi] = 0 (= 4\pi G \rho_0 \boldsymbol{\xi} \cdot \boldsymbol{n}), a_\ell = 0.$	Agreement	Agreement	$M1_{HDG} = M2_{HDG} = M1_{CG} = M2_{CG}$
BVP 2	$[\delta_\phi] = 0;\ [\partial_r \delta_\phi] = 4\pi G \rho_0\, \boldsymbol{\xi} \cdot \boldsymbol{n},\ e_\ell = 0.$			

- HDG is robust.



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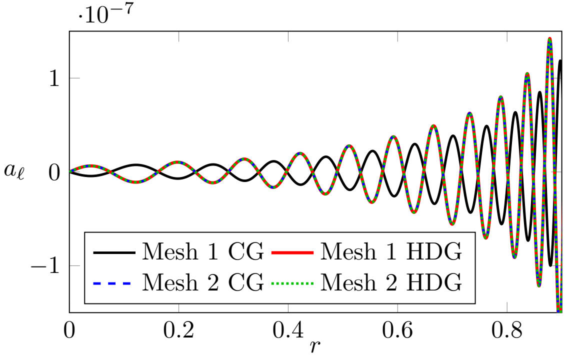
- HDG is robust.

- CG need a more refined around r = r_b where the jump condition different to zero is imposed.



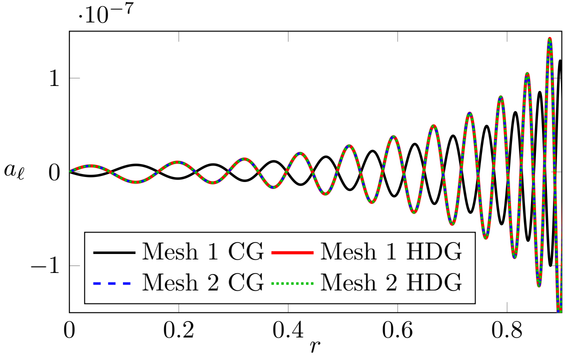
Solution $a_\ell(r, s)$ with Dirac source at $s = 1$ and $\ell = 2$, $\omega/2\pi = 5$ mHz.

Numerical result 2: Comparison HDG vs CG method

		HDG	CG	HDG + CG
BVP 1	$[\delta_\phi] = [\partial_r \delta_\phi] = 0 (= 4\pi G \rho_0 \xi \cdot \boldsymbol{n}), a_\ell = 0.$	Agreement	Agreement	$M1_{HDG} = M2_{HDG} = M1_{CG} = M2_{CG}$
BVP 2	$[\delta_\phi] = 0; \; [\partial_r \delta_\phi] = 4\pi G \rho_0 \, \xi \cdot \boldsymbol{n}, \; e_\ell = 0.$	Agreement	No agreement	$M1_{HDG} = M2_{HDG} = M2_{CG} \neq M1_{CG}$

- HDG is robust.

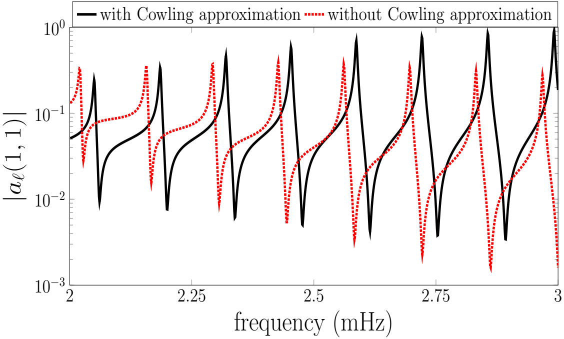
- CG need a more refined around r = r_b where the jump condition different to zero is imposed.



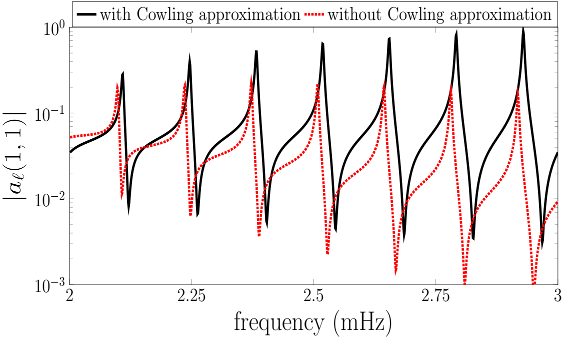
Solution $a_\ell(r, s)$ with Dirac source at $s = 1$ and $\ell = 2$, $\omega / 2\pi = 5$ mHz.

Application : Effect of Cowling's approximation

Superposition of $a_\ell(1, 1; \omega)$ with and without Cowling's approximation.



$\ell = 1.$



$\ell = 5.$

Existence of a shift with Cowling approximation predominant at low mode.

T. P. Larson and J. Schou *Analysis of Medium-ℓ Data from the Michelson Doppler Imager*
Michelson Doppler Imager, Solar Physics, 290 (2015), pp. 3221–3256

Table of Contents

1. Motivations
2. Equations and general question
3. PART I : Wave solver - Resolution in radial symmetry
4. PART II : Spectral solver - Resolution in radial symmetry

Alternative to obtain the system of equations

Unknowns : $\boldsymbol{\xi} = \xi_r e_r + \boldsymbol{\xi}_h, \quad \delta_\phi$

$$\begin{cases} -(\sigma^2 \rho_0 + \mathcal{L})\,\boldsymbol{\xi} \;+\; \nabla \delta_\phi = \mathbf{F}, \\ \Delta \delta_\phi = -4\pi G\, \nabla \cdot (\rho_0 \boldsymbol{\xi}) \,. \end{cases}$$

$$\mathcal{L} = \nabla(\gamma\, p_0 \nabla_{\mathbf{x}} \cdot \xi) - (\nabla p_0)(\nabla_{\mathbf{x}} \cdot \xi) + \nabla[(\xi \cdot \nabla) p_0] - (\xi \cdot \nabla)\nabla p_0 - \rho_0\,(\xi \cdot \nabla_{\mathbf{x}})\,\nabla \phi_0.$$

Coefficients of unknowns in VSH :

$$\boldsymbol{\xi} \leftrightarrow (a_\ell, b_\ell, c_\ell)\,, \qquad \delta_\phi \leftrightarrow d_\ell$$

Alternative to obtain the system of equations

Unknowns : $\boldsymbol{\xi} = \xi_r e_r + \boldsymbol{\xi}_h,$ δ_ϕ & δ_p (Perturbation of pressure)

$$\begin{cases} -(\sigma^2 \rho_0 + \mathcal{L})\,\boldsymbol{\xi} + \nabla \delta_\phi = \mathbf{F}, \\ \Delta \delta_\phi = -4\pi G\, \nabla \cdot (\rho_0 \boldsymbol{\xi}) \,. \end{cases}$$

$$\mathcal{L} = \nabla(\gamma\, p_0 \nabla_{\mathbf{x}} \cdot \boldsymbol{\xi}) - (\nabla p_0)(\nabla_{\mathbf{x}} \cdot \boldsymbol{\xi}) + \nabla[(\boldsymbol{\xi} \cdot \nabla) p_0] - (\boldsymbol{\xi} \cdot \nabla)\nabla p_0 - \rho_0\,(\boldsymbol{\xi} \cdot \nabla_{\mathbf{x}})\,\nabla \phi_0.$$

$$\begin{cases} -\sigma^2 \rho_0\, \boldsymbol{\xi} + \nabla \delta_p + \delta_\rho \nabla \phi_0 + \nabla \delta_\phi = \mathbf{F}, \\ \delta_\rho = -(\nabla \rho_0) \cdot \boldsymbol{\xi} - \rho_0\, \nabla \cdot \boldsymbol{\xi} \\ \delta_p = -\boldsymbol{\xi} \cdot \nabla p_0 - \rho_0 c_0^2 \nabla \cdot \boldsymbol{\xi} \\ \Delta \delta_\phi = -4\pi G\, \nabla \cdot (\rho_0 \boldsymbol{\xi}) \,. \end{cases}$$

Coefficients of unknowns in VSH :

$\boldsymbol{\xi} \leftrightarrow (a_\ell, b_\ell, c_\ell)\,,$ $\delta_\phi \leftrightarrow d_\ell$ & $\delta_p \leftrightarrow e_\ell$

Reminder of unknowns for 1.5D problem

Coefficients of unknowns in Vector spherical harmonics

$$\xi \leftrightarrow (a_\ell, b_\ell, c_\ell)\,, \quad \Big| \quad \delta_\phi \leftrightarrow d_\ell \quad \Big| \quad \delta_p \leftrightarrow e_\ell$$

- *Objective :* obtain a formulation **affine in** σ^2
- **4 unknowns** obtained after Liouville change of variables for the *regular singularities*

$$\tilde{a}_\ell = r\sqrt{\rho_0}\,a_\ell, \quad \tilde{e}_\ell = \frac{\sqrt{\rho_0}}{r}e_\ell, \quad \tilde{d}_\ell = \sqrt{\rho_0}\,d_\ell, \quad \text{and} \quad \tilde{w}_\ell = r\,\tilde{d}_\ell.$$

- **first order** problem

First order eigenproblem

Objective : Find $U = \begin{pmatrix} \tilde{e}_\ell \\ \tilde{d}_\ell \end{pmatrix}$ and $V = \begin{pmatrix} \tilde{a}_\ell \\ \tilde{w}_\ell \end{pmatrix}$ such that

$$\begin{cases} \partial_r U + \mathbb{A}_u U + \mathbb{A}_{uv} V = \sigma^2 \Big(\mathbb{M}_u U + \mathbb{M}_{uv} V \Big), \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_r V + \mathbb{A}_v V + \mathbb{A}_{vu} U \\ \qquad = \sigma^2 \left(\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \partial_r V + \mathbb{M}_v V + \mathbb{M}_{vu} U \right) \end{cases}$$

coupled with BC :

- at $r = 0$, **singular regular**

$$\begin{cases} \dfrac{\alpha_{p_0}}{\gamma} \tilde{e}_\ell - \sigma^2 \tilde{a}_\ell = 0; \\ w_\ell' + (r \dfrac{\alpha_{p_0}}{2} - \ell) \tilde{d}_\ell = 0. \end{cases}$$

- at $r_b = 1.001$,

$$\begin{cases} \rho_0 \tilde{e}_\ell + (\rho_0 \, c_0^2 \mathrm{Ehe} - \rho_0 \Phi_0') \tilde{a}_\ell = 0 & \textbf{zero-surface Lagrangian pressure} \\ [\![\tilde{w}_\ell + \dfrac{\alpha_{\rho_0}}{2} \tilde{d}_\ell]\!] = -4\pi G \rho_0 \tilde{a}_\ell^- & \textbf{Jump condition} \end{cases}$$

- at $r_{\max}$, **exact DtN**

$$\tilde{w}_\ell + (\ell + 1 + r \frac{\alpha_{p_0}}{2}) \tilde{d}_\ell = 0$$

DG method - key points

Local problem **On each element** K_e of the mesh,

$$\begin{cases} -\int_{K^e} U(\partial_r W) + \int_{K^e} (\mathbb{A}_u U) W + \int_{K^e} (\mathbb{A}_{uv} V) W + \int_{\partial K_e} \hat{U} W = \sigma^2 \left(\int_{K_e} (\mathbb{M}_u U) W + \int_{K_e} (\mathbb{M}_{uv} V) W \right), \\ \int_{K_e} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} V(\partial_r W) + \int_{K_e} (\mathbb{A}_v V) W + \int_{K_e} (\mathbb{A}_{vu} U) W + \int_{\partial K_e} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \hat{V} W \\ \qquad = \sigma^2 \left(\int_{K_e} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V(\partial_r W) + \int_{K_e} (\mathbb{M}_v V) W + \int_{K_e} (\mathbb{M}_{vu} U) W + \int_{\partial K_e} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \hat{V} W \right) \end{cases}$$

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with the expression of the **numerical fluxes** in the interior faces

$$\hat{U} = \{U\} + \frac{1}{2} [\![U]\!], \qquad \hat{V} = \{V\} - \frac{1}{2} [\![V]\!] - \begin{pmatrix} \tau_e & 0 \\ 0 & \tau_d \end{pmatrix} [\![U]\!]$$

and adapted numerical fluxes on the exterior faces.
We made a reverse integration by part on the first equation and then ...

DG method - key points

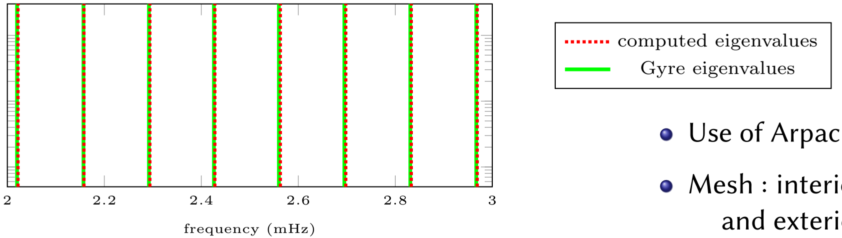
Global problem **Sum on all elements** of the mesh.

$$\begin{cases} \int_\Omega (\partial_r U) W + \int_\Omega (\mathbb{A}_u U) W + \int_\Omega (\mathbb{A}_{uv} V) W - s_h(U, W) = \sigma^2 \left(\int_\Omega (\mathbb{M}_u U) W + \int_\Omega (\mathbb{M}_{uv} V) W \right), \\ \int_\Omega \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} V (\partial_r W) + \int_\Omega (\mathbb{A}_v V) W + \int_\Omega (\mathbb{A}_{vu} U) W + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (s_h(W, V) - \tau(U, W)) + \int_{\Sigma_B} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \hat{V} W \\ \quad = \sigma^2 \left(\int_\Omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V (\partial_r W) + \int_\Omega (\mathbb{M}_v V) W + \int_\Omega (\mathbb{M}_{vu} U) W + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} (s_h(W, V) - \tau(U, W)) + \int_{\Sigma_B} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \hat{V} W \right) \end{cases}$$

$$\text{with} \quad s_h(U, W) = \int_{\Sigma_I} \left(\{W\} - \frac{1}{2} [\![W]\!] \right) \cdot [\![U]\!], \qquad \tau(U, W) = \int_{\Sigma_I} \begin{pmatrix} \tau_e & 0 \\ 0 & \tau_d \end{pmatrix} ([\![U]\!] \cdot [\![W]\!])$$

.

Numerical results 3 : LDG eigensolver



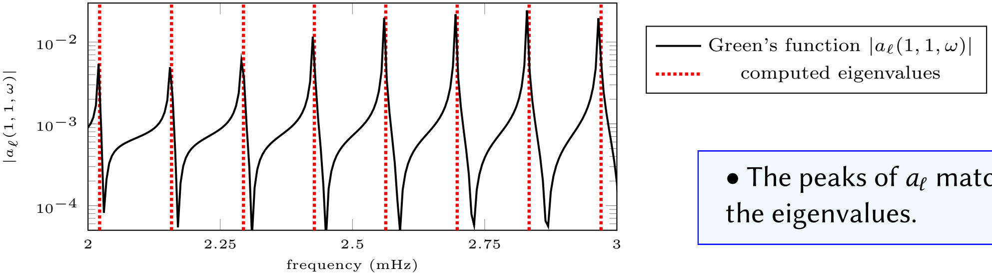
- Use of Arpack solver.
- Mesh : interior size $h_1 = 1e - 03$ and exterior size $h_2 = 5e - 04$.

Comparisons of eigenvalues obtained by Gyre solver and by our eigensolver at $\ell = 1$.

Our eigenvalues match with EV obtained with Gyre solver.

RHD Townsend and SA Teitler *Gyre : an open-source stellar oscillation code based on a new magnus multiple shooting scheme*
Monthly Notices of the Royal Astronomical Society,2013

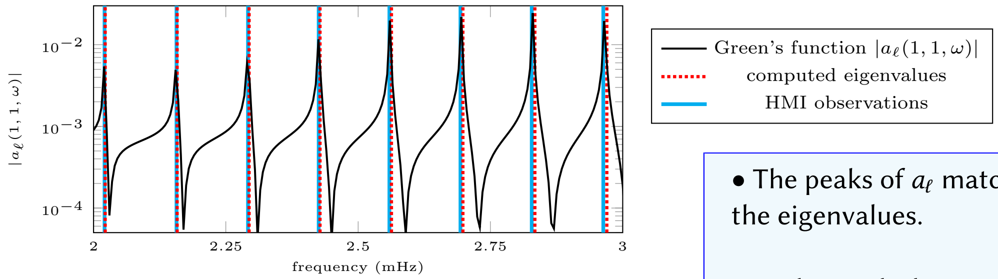
Comparison between eigensolver and GK solver



Superposition of Green kernel of the wave problem and eigenvalues at $\ell = 1$.

- The peaks of a_ℓ match with the eigenvalues.

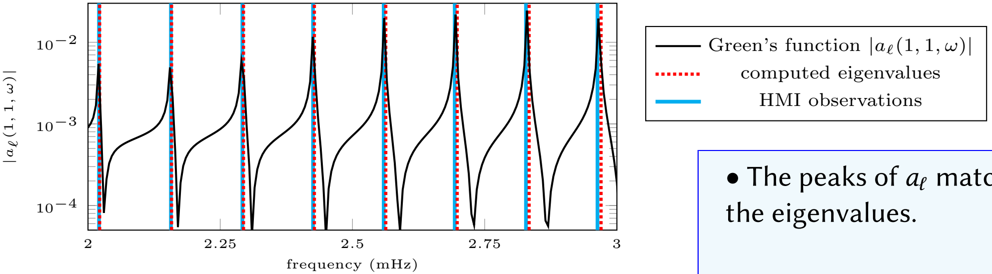
Application : Validation by comparisons with HMI observables



Superposition of solutions of the wave problem and eigenproblem at $\ell = 1$.

- The peaks of a_ℓ match with the eigenvalues.
- Both match the HMI EV.
- Validation of HDG wave problem solver and LDG eigensolver.

Application : Validation by comparisons with HMI observables



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- The peaks of a_ℓ match with the eigenvalues.
- Both match the HMI EV.
- Validation of HDG wave problem solver and LDG eigensolver.

Summary / perspectives

- We have built a computation framework employing the HDG and CG method without Cowling approximation to compute the Green's kernel.
  - HDG method needs appropriate stabilization parameters.
  - CG method not suited for BVP 2 with BC $\delta_p = 0$.
  - Cowling's approximation generates a phase shift in solutions which is predominant at low modes.
- For the LDG eigensolver, our eigenvalues match with Gyre ones and then with Green's Kernel.
- Removing Cowling approximation, our simulations have good correspondance with HMI observables.

Thank you.

Zero-pressure surface condition for a_ℓ at the surface

Perturbation to pressure

$$\delta_p := -\xi \cdot \nabla p_0 - \rho_0 c_0^2 \nabla \cdot \xi$$

$$\delta_p = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} e_\ell^m Y_\ell^m.$$

$$\delta_p = 0 \iff e_\ell^m = 0,$$

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$$\delta_p = 0 \iff e_\ell^m = 0,$$

Coefficients e_ℓ and a_ℓ (of radial ξ_r) are related (in scaled coordinates) by

$$e_\ell = -\sigma^2 \frac{\rho_0 c_0^2}{(\sigma^2 - S_\ell^2)} \left[a_\ell' + \left(\frac{2}{r} - \frac{\alpha_{p_0}}{\gamma} \right) a_\ell + \frac{\sqrt{\ell(\ell+1)}}{\sigma^2 r \rho_0} g_\ell^m \right].$$

$\alpha_{p_0} = -\frac{p_0'}{p_0}$

σ complex frequency containing attenuation,

$S_\ell^2 = \ell(\ell+1)\frac{c_0^2}{r^2 L_0^2}$ Lamb frequency.

Zero-pressure surface condition for a_ℓ at the surface

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$$\delta_p = 0 \quad \Leftrightarrow \quad e_\ell^m = 0,$$

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σ complex frequency containing attenuation,

$S_\ell^2 = \ell(\ell+1)\frac{c_0^2}{r^2 L_0^2}$ Lamb frequency.

Assume $g_\ell^m \equiv 0$ near $r = r_b$:

$$e_\ell = 0 \quad \underset{\text{at } r=r_b}{\Leftrightarrow} \quad r a_\ell' + \left(2 - \frac{\alpha_{p_0}}{\gamma} r\right) a_\ell = 0 .$$

D-t-N boundary condition for δ_ϕ at $r_{\max} = r_b + \epsilon$

Assumption: $\rho_0 = 0$ for $r > r_b$.

Goal : Compute an artificiel boundary condition at , $r = r_{\text{abc}}$ with $r_{\max} > r_b = 1.001$

Main ideas for derivation of DtN

Define $\mathbb{B}^+ := \{|\mathbf{x}| > r_{\max}\}$ and $\delta_\phi^+ := \delta_\phi|_{\mathbb{B}^+}$.

- On $\mathbb{B}^+$, δ_ϕ^+ satisfies the Laplace equation,

$$\Delta\delta_\phi^+ = 0\ , \quad \text{and } \delta_\phi^+ \to 0 \text{ as } r = |\mathbf{x}| \to \infty \ (*).$$

- General solutions of (*) have the form,

$$\delta_\phi^+ = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{\mathfrak{d}_\ell^{m+}}{r^{\ell+1}} Y_\ell^m(\hat{\mathbf{x}}),$$

$$\Rightarrow \partial_r\delta_\phi^+ = -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{(\ell+1)\mathfrak{d}_\ell^{m+}}{r^{\ell+2}} Y_\ell^m(\hat{\mathbf{x}}).$$

- $\delta_\phi^+ = \delta_\phi^-$ is harmonic on $\mathbb{B}^+$, in particular is H^1 in small neighborhood of $r = r_{\max}$,

$$\begin{cases} \delta_\phi^- = \delta_\phi^+ \\ \partial_r\delta_\phi^- = \partial_r\delta_{\phi^+} \end{cases} \underset{(\ell,m)}{\overset{\text{for each}}{\Longrightarrow}} \begin{cases} d_\ell^- = \dfrac{\mathfrak{d}_\ell^{m+}}{r^{\ell+1}}, \\ \partial_r d_\ell^- = -\dfrac{(\ell+1)\mathfrak{d}_\ell^{m+}}{r^{\ell+2}} \end{cases}, \text{ at } r = r_{\max},$$

$$\Rightarrow \boxed{\partial_r d_\ell + \frac{\ell+1}{r_S} d_\ell = 0 \quad \text{at} \quad r = r_{\max}}$$

Chaljub, E., & Valette, B. (2004). *Spectral element modelling of three-dimensional wave propagation in a self-gravitating Earth with an arbitrarily stratified outer core.*
Geophysical Journal International.

Numerical investigation 1 : HDG method and penalization

Goal : Compute the most physically adapted parameter in stabilization HDG method for 1.5D.

Main ideas for the choice of the stabilization parameter

For Helmholtz equation $(-\Delta - \kappa^2)u = f$ has stabilization

$$\widehat{\nabla u} \cdot \mathbf{n} = \nabla u_h \cdot \mathbf{n} + \mathrm{i}\,\kappa\,\tau\,(\lambda - u_h) \quad \text{with}$$

$$\tau = \mathrm{O}(1) \quad \text{and} \quad \begin{cases} \mathrm{Im}\,\kappa = 0 \; : & \tau > 0\,, \qquad \text{or} \\ \mathrm{Im}\,\kappa \neq 0 \; : & (\mathrm{Re}\,\tau)(\mathrm{Im}\,\kappa) \leq 0 \end{cases}$$

For our case, a definition for the numerical Neumann trace for a_ℓ and d_ℓ have to be given :

$$\boxed{\widehat{r\partial_r a} = r\partial_r a_h + \tau_a(a_h - \lambda_a) \quad \text{and} \quad \widehat{r\partial_r d} = r\partial_r d_h + \tau_d(d_h - \lambda_d)\,,} \tag{2}$$

with stabilization factors denoted τ_a, τ_d.

Cui, Jintao & Zhang. (2014). *An analysis of HDG methods for the Helmholtz equation.* IMA Journal of Numerical Analysis. IMA Journal of Numerical Analysis.

Gopalakrishnan J., Lanteri S., Olivares N. & Perrussel R. (2015). *Stabilization in relation to wavenumber in HDG methods.* Advanced Modeling and Simulation in Engineering Sciences.

Principle of the method

1. *Local problem* on each element $I \subset [0, r_b]$ and $\tilde{I} \subset [r_b, r_{\max}]$

- Unknowns $(a_h, \underbrace{r a_h'}_{v_h}, d_h, \underbrace{r d_h'}_{w_h})$

- **First order mixed** formulation with unknowns coupled with **Dirichlet BC**.

$$
\begin{cases}
r\,\partial_r\, v_h + \alpha_1\, v_h + \alpha_2\, a_h + \alpha_3 w_h + \alpha_4\, d_h = \mathfrak{f} & \text{on } I = [r_i, r_{i+h}];\\
r\partial_r\, w_h + \beta_1\, w_h + \beta_2\, d_h + \beta_3 v_h + \beta_4\, a_h = 0 & \text{on } I = [r_i, r_{i+h}];\\
\beta_1 \partial_r\, w_h + \beta_2\, w_h + \beta_3\, d_h = 0 & \text{on } \tilde{I} = [r_i, r_{i+h}];\\
\\
a_h := \lambda_a & \text{on } \partial_I;\\
d_h := \lambda_d & \text{on } \partial_I \cup \partial_{\tilde{I}};
\end{cases}
$$

Principle of the method for BVP 2

1. *Global problem and relation for numerical traces* $\hat{a}_h^{+}_{-} = \lambda_a$ and $\hat{d}_h^{+}_{-} = \lambda_d$

Continuity condition

For any test function ϕ, with $\mathbf{n}_{\mathfrak{f}}$ the normal on the face

- Interior nodes, $\forall \mathfrak{f} \in \Sigma_I$,

$$\int_{\mathfrak{f}} [\underbrace{\widehat{r\,\partial_r a_h}}_{\widehat{v_h}} \cdot \mathbf{n}_{\mathfrak{f}}]\,\overline{\phi}\, d\mathfrak{f} = 0, \quad \forall \mathfrak{f} \in (0, r_b) \qquad \begin{cases} \displaystyle\int_{\mathfrak{f}} [\underbrace{\widehat{r\,\partial_r d_h}}_{\widehat{w_h}} \cdot \mathbf{n}_{\mathfrak{f}}]\,\overline{\phi}\, d\mathfrak{f} = 0 \quad \forall \mathfrak{f} \in (0, r_b) \cup (r_b, r_{\max}). \\ [\widehat{w_h} \cdot \mathbf{n}_{\mathfrak{f}}] = -4\pi G r \rho_0 a_h \cdot \mathbf{n}_{\mathfrak{f}} \quad \text{at } \mathfrak{f} = r_b. \end{cases}$$

<u>Essence of the HDG method</u> : the formulation of the numerical fluxes

$$\underbrace{\widehat{r\partial_r a_h}^{(e)}}_{\widehat{v_h}^{(e)}} = \underbrace{r\partial_r a_h^{(e)}}_{v_h^{(e)}} + \tau_a (a_h^{(e)} - \lambda_a^{(e)})\, \boldsymbol{n}_{\mathfrak{f}}^{(e)} \quad \text{and} \quad \underbrace{\widehat{r\partial_r d_h}^{(e)}}_{\widehat{w_h}^{(e)}} = \underbrace{r\partial_r d_h^{(e)}}_{w_h^{(e)}} + \tau_d (d_h^{(e)} - \lambda_d^{(e)})\, \boldsymbol{n}_{\mathfrak{f}}^{(e)}, \tag{3}$$

Principle of the method for BVP 2

1. *Global problem and relation for numerical traces* $\hat{a}_h^{\pm} = \lambda_a$ and $\hat{d}_h^{\pm} = \lambda_d$

Continuity condition

- Boundary nodes, $\forall \mathfrak{f} \in \Sigma_B$, only one side remains

$$\boxed{\widehat{v_h}|_{\mathfrak{f}} = v_h + \tau_a(a_h - \lambda_a)\mathbf{n}_{\mathfrak{f}} \quad \text{and} \quad \widehat{w_h}|_{\mathfrak{f}} = w_h + \tau_d(d_h - \lambda_d)\mathbf{n}_{\mathfrak{f}}.} \tag{4}$$

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$$\begin{cases} \widehat{w_h} \cdot \mathbf{n}_{\mathfrak{f}} = -\ell\,\lambda_d = w_h \cdot \mathbf{n}_{\mathfrak{f}} + \tau_d(d_h - \lambda_d) \quad \text{at } r = 0 \\ \widehat{w_h} \cdot \mathbf{n}_{\mathfrak{f}} = -(\ell + 1)\,\lambda_d = w_h \cdot \mathbf{n}_{\mathfrak{f}} + \tau_d(d_h - \lambda_d) \quad \text{at } r = r_{\max}. \end{cases}$$

Result 1 : HDG method and choice of the stabilization parameter

Choice 1 :

$$\tau_a = \tau_d = 1;$$

Note $\nu = 1$ for right end-point of an interval and $\nu = -1$ for left end-point.

Choice 2 ($\star$) :

$$\tau_a = -r\left(\mathfrak{J}_\ell'/\mathfrak{J}_\ell - \mathrm{i}\sqrt{-V_\ell}\,n\right);$$

$$\tau_d = (1 - n\,r)\,,\;\; (\ell = 0) \quad \text{and} \quad \tau_d = 1 - \sqrt{\ell(\ell+1)}\,n,\;\; (\ell > 0).$$

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Cui Jintao and Zhang Wujun (2014).
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