# Outline



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# I won't speak of this

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# But of that

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# Propagation[1] of water waves from the ocean to a harbor

E. Audusse, O. Lafitte, C Sulem

LAGA, Université Sorbonne Paris Nord, Villetaneuse, France IRL CRM-CNRS, Centre de Recherches Mathématiques, Université de Montréal, Canada, Université de Toronto, Canada

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[1] with Michael Levitin, Pierrick Quemar, Nilima Nigam

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High frequency parameter?!?

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# Infinite ocean, flat bottom: find a wave

Undisturbed ocean: $\Omega = \{x \in (-\infty, \infty), z \in (-h_0, 0)\}$.
Equations: $\mathrm{div}\vec{u} = 0$ in $\Omega$, boundary conditions $\vec{u}.n = 0$ at $z = -h_0$.
Potential fluid: $\vec{u}(x, z, t) = \nabla\phi(x, z, t)$.
Perturbation: a free surface appears: $z = \eta(x, t)$. System
($\phi_0(x, t) := \phi(x, 0, t)$, linearization of $\phi(x, \eta, t)$ for $\eta$ small)

$$\begin{cases} \partial_t \phi_0 + g\eta = 0 & (i) \text{ Linearized Bernouilli} \\ \partial_t \eta - \partial_z \phi(x, 0, t) = 0 & (ii) \text{ displacement of the surface} \\ \Delta\phi = 0 & (iii) \text{ incomp. irr.} \\ \partial_z \phi(x, -h_0, t) = 0 & (iv) \end{cases}$$

From the elliptic problem (iii)-(iv) one gets an hyperbolic problem on the surface (i)-(ii) through a Dirichlet to Neumann operator.

# Resolution

- Assume $\phi_0$ given. The problem (iii)-(iv) has a unique solution[2] (valid even in infinite domain), hence there exists an operator $\Lambda$ such that $\partial_z \phi(x, 0, t) = \Lambda(\phi_0)(x, t)$.

[2] Still valid when the bottom is $z = -h_0(x)$, or when the domain is $\Omega_{left} = \{x \in (-\infty, 0), z \in (-h_0, 0)\}$+Neumann condition at $x = 0$.

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• Partial Fourier transform (after symmetrization if $\Omega_{left}$):

$$\frac{d^2}{dz^2}\hat{\phi}(k,z,t) = k^2\hat{\phi}(k,z,t) \Rightarrow$$
$$\hat{\phi}(k,z,t) = A(k,t)\cosh kz + B(k,t)\sinh kz$$

From $\hat{\phi}(k,0,t) = \hat{\phi}_0(k,t), \hat{\phi}'(k,-h_0,t) = 0$,

$$\hat{\phi}(k,z,t) = \hat{\phi}_0(k,t)\frac{\cosh k(z+h_0)}{\cosh kh_0}.$$

Obtain $\frac{d}{dz}\hat{\phi}(k,0,t) = k\tanh kh_0\hat{\phi}_0(k,t)$:

Dirichlet to Neumann Fourier multiplier.

[2] Still valid when the bottom is $z = -h_0(x)$, or when the domain is $\Omega_{left} = \{x \in (-\infty, 0), z \in (-h_0, 0)\}$+Neumann condition at $x = 0$.

Replace in (i), (ii):

$$\begin{cases} \partial_t \hat{\phi}_0(k,t) + g\hat{\eta}(k,t) = 0 \\ \partial_t \hat{\eta}(k,t) = k \tanh kh_0 \hat{\phi}_0(k,t) \end{cases}$$

$$\Rightarrow \begin{cases} \hat{\eta}(k,t) = -\frac{1}{g}\hat{\phi}_0(k,t), \\ \partial^2_{t^2}\hat{\phi}_0(k,t) + gk \tanh kh_0 \hat{\phi}_0(k,t) = 0 \end{cases}$$

Solutions $\hat{\phi}_0(k,t) = a(k)e^{i\sqrt{gk \tanh kh_0}\,t} + b(k)e^{-i\sqrt{gk \tanh kh_0}\,t} \Rightarrow \phi_0(x,t) = \check{\hat{a}} \star E_+(x,t) + \check{\hat{b}} \star E_-(x,t)$, with (oscillatory integral)

$$E_\pm(x,t) = \frac{1}{2\pi}\oint e^{i(kx \pm t\sqrt{gk \tanh kh_0})}dk.$$

After Fourier transform in $t$,
$\hat{E}_+(x,\omega_0) = \frac{1}{2\pi}\oint\oint e^{i(kx+t\sqrt{gk\tanh kh_0}-\omega_0 t)}dtdk.$
Let

$$k_0 > 0 \text{ solution of } \sqrt{gk_0\tanh k_0 h_0} = \omega_0.$$

Jacobian:
$(k-k_0)v(k,k_0) = \sqrt{gk\tanh kh_0} - \omega_0,\, v_0 = \frac{1}{2}\sqrt{\frac{g}{k_0}}\frac{\tanh k_0 h_0 + kh_0\cosh^{-2}k_0 h_0}{\sqrt{\tanh k_0 h_0}}.$
Use $\frac{1}{2\pi}\oint e^{itv(k,k_0)(k-k_0)}dt = \frac{1}{v(k_0,k_0)}\delta_{k_0}$ to obtain, for $\omega_0 > 0$

$$\hat{E}_+(x,\omega_0) = v_0^{-1}e^{ik_0 x}, \hat{E}_-(x,-\omega_0) = v_0^{-1}e^{ik_0 x}.$$

Fourier mode in time in $\omega_0, -\omega_0$:

$$[a(k_0)\cos(k_0 x + \omega_0 t) + b(k_0)\cos(k_0 x - \omega_0 t)]\frac{\cosh k_0(z+h_0)}{v_0\cosh k_0 h_0}.$$

$\phi_0$, $\eta$, $\phi$ are waves.

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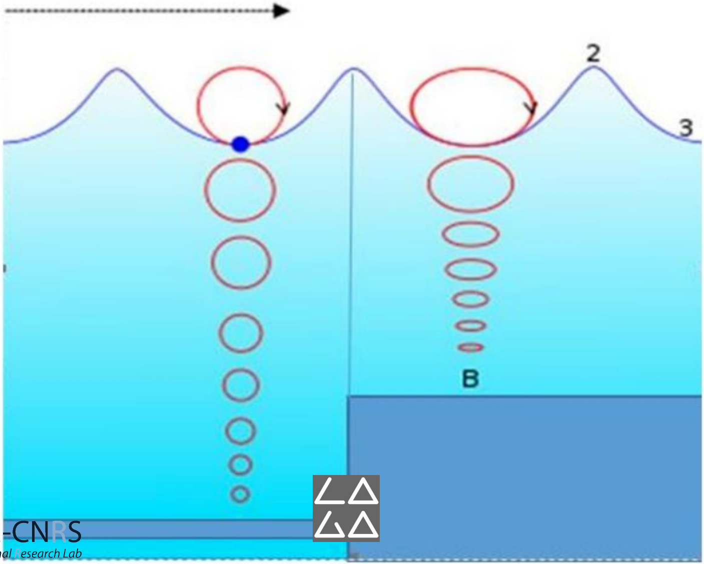
# Movement of the particles

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# Non flat bottom

Domain is $\Omega = \{(x,z) \in \mathbb{R}_- \times \mathbb{R}, -h(x) \leq z \leq 0\}$. Introduce $\tilde{\Omega} = \{(x,z) \in \mathbb{R} \times \mathbb{R}, -h(|x|) \leq z \leq 0\}$.
Assume there exists $h_0 > 0$ such that $b := h - h_0$ is compactly supported, with $a$: length of the support of $b = h - h_0$.
Physical problem (harbor), $\Psi$ a function on $\Omega \times \mathbb{R}$ (linearized Euler for the free surface equation):

$$\begin{cases} \Delta\Psi = 0 \\ \partial_n \Psi(x, -h(x), t) = 0, \partial_x \Psi(0, z, t) = 0 \end{cases}$$

Free surface system:

$$\begin{cases} \partial_t \Psi(x, 0, t) + g\eta(x,t) = 0 \\ \partial_t \eta - \partial_z \Psi(x, 0, t) = 0 \end{cases}$$

# Dirichlet to Neumann operator

Problem on $\Phi$ (on $\tilde{\Omega}$):

$$\begin{cases} \Delta\Phi = 0, \tilde{\Omega} \\ \partial_n \Phi = 0, \text{ on } z = -h(x) \end{cases} \tag{1}$$

Problem on $\Phi_*$ on $\Omega$: identical, adding $\partial_x \Phi_*(0, z) = 0$ on $x = 0$.

## Proposition

*For $\phi \in \dot{H}^{\frac{1}{2}}(\mathbb{R})$, the problem* $\begin{cases} \Phi(x,0) = \phi(x) \\ \Delta\Phi = 0 \\ \partial_n\Phi = 0, z = -h(x) \end{cases}$ *has a unique solution in $\dot{H}^1(\tilde{\Omega})$ (variational problem with inhomogeneous Dirichlet condition on top). Operator $\Lambda$: $\dot{H}^{\frac{1}{2}}(\mathbb{R}) \to \dot{H}^{-\frac{1}{2}}(\mathbb{R})$: $\phi \to \partial_z\Phi(.,0)$: pseudo-differential operator (DTN). If $\phi$ even, the restriction of $\Phi$ on $\tilde{\Omega}$ satisfies (1) and DTN similar.*

# Wave equation system

Determination of $\phi_\lambda$ associated with the eigenvalue $\lambda$ of the DTN.

$$\begin{cases} \partial_t \phi + g\eta = 0 \\ \partial_t \eta - \Lambda \phi = 0 \end{cases}$$

Spectral projection on an eigenmode $\phi_\lambda$ associated with an eigenvalue $\lambda$ of the Dirichlet to Neumann $\phi := C \otimes \phi_\lambda$:

$$\begin{cases} C'(t)\phi_\lambda + g\eta = 0 \\ C''(t) + \lambda g C(t) = 0 \end{cases}$$

# Expansion of the DTN as a pseudodifferential operator

(Craig-Sulem..) Notations $D = Op(k) = \frac{1}{i}\frac{d}{dx}$, $b(x) = h_0 - h(x)$.
Expression of the solution of the Laplace equation:

$$\Phi(x,z) = [\frac{\cosh(z+h_0)D}{\cosh h_0 D}](\phi)(x) + [\sinh zD](\psi)(x). \qquad (2)$$

Define the operators $A(b)$ and $C(b)$ through
$(A(b)\phi)(x) = \int e^{ikx}\frac{\sinh kb}{\cosh kh_0}\hat{\phi}(k)dk, (C(b)\phi)(x) = \int e^{ikx}\cosh k(h_0 + b(x))\hat{\phi}(k)dk.$

The equation $(\partial_z - b'\partial_x)\Phi = 0$ on $z = -h(x)$ rewrites $A(b)\phi + C(b)\psi = 0$ and the DTN operator is given by $DTN(\phi) = D\tanh h_0 D(\phi) + D\psi$.

Proposition (C-S 2005): There exists an operator $L(b)$ such that

$$\Phi(x,z) = [\frac{\cosh(z+h_0)D}{\cosh h_0 D}](\phi)(x) + [\sinh zD](L(b)\phi)(x). \quad (3)$$

The operator $L(b)$ is given by $-B(b)A(b)$, with $B(b) = C(b)^{-1}$.

The Dirichlet to Neumann operator is thus
$DTN(\phi) = [D \tanh h_0 D](\phi) + DL(b)(\phi)$.

Expansion of $L(b)$ (in powers of $b$) can be found in the annex of (C-S 2005): $L_1 = -\frac{1}{\cosh h_0 k} \star b \star \frac{1}{\cosh h_0 k}, -L_2 = \frac{1}{\cosh h_0 k} \star Op(bk) \star \sinh h_0 k \star L_1 \ldots$

# Approximate equations: Helmholtz, SW, Berkhoff

•Shallow water regime corresponds to the equation

$$P_{SW}(l) := \frac{d}{dx}(h(x)\frac{dl}{dx}) + \lambda l(x) = 0, l'(0) = 0. \quad (4)$$

•Deep water regime ($kh_0 >> 1$) : Let $k_\lambda(x)$ unique positive solution of $k_\lambda(x)\tanh(k_\lambda(x)h(x)) = \lambda$: Helmholtz equation ($k_\lambda \simeq cst$)

$$P_{DW}(\psi) := \psi''(x) + (k_\lambda(x))^2\psi(x) = 0, \psi'(0) = 0. \quad (5)$$

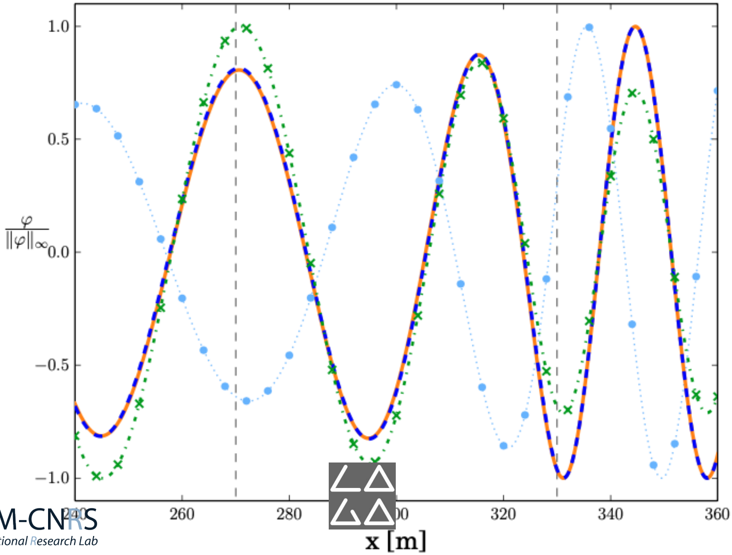
•Intermediate regime: Let introduce $T_\lambda$ a function[3] constructed from $\lambda$ and $h$, one obtains the equation

$$P_{IR}(\phi) := \frac{d}{dx}(T_\lambda(x)\frac{d\phi}{dx}) + (k_\lambda(x))^2 T_\lambda(x)\phi(x) = 0, \phi'(0) = 0, \quad (6)$$

introduced by Eckart (1952), Berkhoff (1976) as the mild slope eq.

[3] such that $2k_\lambda(x)T_\lambda(x)(\cosh(k_\lambda(x)h(x)))^2 = k_\lambda(x)h(x) + \frac{\sinh 2k_\lambda(x)h(x)}{2}$

# Numerical comparison between solutions

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# Geometrical optics and physical optics

Geometrical optics approximation (at frequency 1):. Model: action of the differential operator $D^3$ on $ae^{i\theta}$:

$$\begin{aligned}(\tfrac{d}{dx})^3(ae^{i\theta}) &= e^{i\theta}\Big[a''' + 3i\theta' a'' + 3a'[-(\theta')^2 + i\theta''] \\ &+a[-3\theta''\theta' - i(\theta')^3 + i\theta''']\Big]\end{aligned}$$

Reordering (lowest to highest order) and creating a **hierarchy** of terms:

$$\begin{aligned}e^{-i\theta}(\tfrac{d}{dx})^3(ae^{i\theta}) &= a''' + i[3\theta' a'' - 3\theta'' a' + \theta''' a] \\ &+[-3\theta''\theta' a - (\theta')^2 a'] - i(\theta')^3 a.\end{aligned}$$

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Finding the geometrical optics solution of $PU = 0$ amounts to put to zero the two highest order terms of $P(ae^{i\theta})$.

Geometrical optics method used for the three approx. models

SW: $I = a_{SW}e^{i\theta_0} \to (\theta_0')^2 = \frac{\lambda}{h(x)}, a_{SW}^2\theta_0' h = cst = a_{SW}^2\sqrt{\lambda h}$.

DW: $\theta'(x) = k_\lambda(x)$, $\psi = a_{DW}e^{i\theta} \to 2\theta' a_{DW}' + \theta'' a_{DW} \to a_{DW}\sqrt{\theta'} = cst$.

IR: $\theta'(x) = k_\lambda(x)$, $\psi = a_{IR}e^{i\theta} \to a_{IR}^2 T_\lambda k_\lambda = cst$.

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**SW: obtain** $i(2h\theta_0' a_{SW}' + (h\theta_0')' a_{SW}) + (h a_{SW}')'$ which gives $a_{SW} = h^{-\frac{1}{4}}$ hence $(h a_{SW}')' = (-\frac{1}{4} h^{-\frac{1}{4}} h')' = h^{-\frac{5}{4}}[-\frac{1}{4} h'' h + \frac{1}{16}(h')^2]$.

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hence $(T_\lambda a_{IR}')' = (T_\lambda f'(b) b')' = T_\lambda f'(b) b'' + \partial_b (T_\lambda f')(b')^2$.

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hence $(T_\lambda a_{IR}')' = (T_\lambda f'(b) b')' = T_\lambda f'(b) b'' + \partial_b(T_\lambda f')(b')^2$.

**For all three models, one has** $P_M(a_M e^{i\theta_M}) = O(b'' + (b')^2)$.

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# Formal 'asymptotics' expansion of a PDO

Aim: write the action of a pseudodifferential operator $Op(A)$ on $x \to a(x)e^{i\theta(x)}$: $Op(A(b(x), D))(a(x)e^{i\theta(x)}) \simeq \frac{1}{2\pi}\int e^{i(x-x')k}A(b(x),k)a(x')e^{i\theta_\lambda(x')}dx'dk.$

## Definition

*We call formal asymptotics expansion of $Op(A)$ on $x \to a(x)e^{i\theta(x)}$ at order $N$ the sequence*

$$Op(A)^{(N)}(ae^{i\theta}) = [A(b(x),\theta'(x))(\sum_{j=0}^{N} a_j(x)) + \sum_{j=1}^{N} L_j^{A,\theta'}(a)]e^{i\theta(x)}$$

*where $L_j^{A,\theta'}$ is the $j-th$ term in Hörmander. With $A^{(l)} = \partial^l_{k^l}A(b(x),k)$,*

$L_1^{A,\theta'}(a) = A'(b(x),\theta')a' + \frac{1}{2}\theta''A''(b(x),\theta')a,$

$L_2^{A,\theta'}(a) = A''(b(x),\theta')a'' + \theta''A^{(3)}(b(x),\theta')a' + \frac{1}{4}(\theta'')^2A^{(4)}(b(x),\theta')a + \frac{1}{6}\theta'''A^{(3)}(b(x),\theta')a.$

Hint of the proof: replace $Op(A)$ by its Taylor polynomial. Observe that

$$Op((k-\theta'(x))^j)(ae^{ik\theta}) = \frac{1}{2\pi}\oint \quad \mathcal{F}[a(x+u)e^{\frac{i}{2}\theta''(x)u^2-i\eta u}R(u)]d\eta$$

# Hierarchical system for a general operator

## Definition

One says that $a := a_0 + a_1 + \cdots a_N$, $a_j(0) = 0$ for $j \geq 1$, is a solution of the formal asymptotic expansion of $Op(P)(ae^{i\theta}) = \lambda a e^{i\theta} + (\sum_{j=0}^{N} f_j)e^{i\theta}$ if

$$\left\{ \begin{array}{l} P(b(x), \theta')a_0 = \lambda a_0 + f_0, \\ L_1^P(a_0) + P(b(x), \theta')a_1 = \lambda a_1 + f_1, \\ \sum_{j=0}^{p-1} L_j^P(a_{p-j}) + P(b(x), \theta')a_p = \lambda a_p + f_p. \end{array} \right. \tag{7}$$

## Proposition

*If $\theta', \lambda$ does not solve $det(P(b(x), \theta') - \lambda) = 0$, this system has no solution. Otherwise, this system has a solution if $Ker(P(b(x), \theta') - \lambda)$ and $Im(P(b(x), \theta') - \lambda)$ are supplementary spaces.*

# Fundamental equation for the Craig-Sulem system:

Eigenvector of the DTN operator of CS 2005 $\Leftrightarrow$ zero solution of

$$\begin{cases} Op(A(b,k))\phi + Op(C(b,k))\psi = 0, \\ Op(D\tanh h_0 D)\phi + \psi' = \lambda\phi. \end{cases}$$

$$Op(A)(ae^{i\theta}) = A(b(x),\theta'(x))a + \frac{1}{i}\left[\frac{1}{2}\theta''\frac{\partial^2 A(b,k)}{\partial k^2}a + \frac{\partial A(b,k)}{\partial k}a'\right] + l.o.t.$$

System (defining $G_0(k) := k\tanh kh_0$, Fourier multiplier):

$$\begin{cases} Aa + Cb + \frac{1}{i}[A'a' + \frac{1}{2}\theta'' A'' a + C'b' + \frac{1}{2}\theta'' C'' b] = l.o.t. \\ G_0 a + \theta' b + \frac{1}{i}[G_0' a' + \frac{1}{2}\theta'' G_0'' + b'] - \lambda a = l.o.t \end{cases}$$

Hierarchical system (similar to the high frequency expansion)

$$\begin{cases} Aa_0 + Cb_0 = 0 \\ G_0 a_0 + \theta' b_0 - \lambda a_0 = 0 \end{cases}$$

$$\begin{cases} Aa_1 + Cb_1 + \frac{1}{i}[A'a_0' + \frac{1}{2}\theta'' A'' a_0 + C'b_0' + \frac{1}{2}\theta'' C'' b_0] = 0. \\ G_0 a_1 + \theta' b_1 + \frac{1}{i}[G_0' a_0' + \frac{1}{2}\theta'' G_0'' a_0 + b_0'] - \lambda a_1 = 0, \ldots \end{cases}$$

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First relation (eikonal equation):

$$(G_0(\theta') - \lambda)C(b(x), \theta') - \theta' A(b(x), \theta') = 0 \Leftrightarrow \theta' \tanh h(x)\theta' = \lambda. \quad (8)$$

Let $\mathcal{T}$ be the reciprocal function of $X \to X \tanh X$.

$$\theta'(x) = \frac{\mathcal{T}(\lambda h(x))}{h(x)}. \quad (9)$$

Main observation: $\lambda h$ close to 0 yields $\theta'(x) \simeq \sqrt{\frac{\lambda}{h}}$, $\lambda h$ large yields $\theta'(x) \simeq \lambda$.

Intermediate regime: $\lambda h \in [\frac{1}{2} \tanh \frac{1}{2}, \frac{5}{2} \tanh \frac{5}{2}] = [0.23, 2.47]$.

# Geometrical optics (as JMR)

Define $M_0(b(x),\theta') = \begin{pmatrix} A(b(x),\theta') & C(b(x),\theta') \\ G_0(\theta') - \lambda & \theta' \end{pmatrix}$.

- Equation $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \in \mathrm{Ker} M_0$
- Equation $\begin{pmatrix} \frac{1}{i}[A' a_0' + \frac{1}{2}\theta'' A'' a_0 + C' b_0' + \frac{1}{2}\theta'' C'' b_0] \\ \frac{1}{i}[G_0' a_0' + \frac{1}{2}\theta'' G_0'' a_0 + b_0'] \end{pmatrix} \in \mathrm{Im} M_0$
- By combination, ordinary differential equation on $a_0$.

Indeed $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = a_0 \begin{pmatrix} 1 \\ -\frac{A(b(x),\theta')}{C(b(x),\theta')} \end{pmatrix} = a_0 \begin{pmatrix} 1 \\ -\frac{G_0(\theta')-\lambda}{\theta'} \end{pmatrix}$ and

$-\theta'[A' a_0' + \frac{1}{2}\theta'' A'' a_0 + C' b_0' + \frac{1}{2}\theta'' C'' b_0] + C(b(x),\theta')\frac{1}{i}[G_0' a_0' + \frac{1}{2}\theta'' G_0'' a_0 + b_0'] = 0.$

Equivalent to

$$E(b(x),\theta')\frac{a_0'}{a_0} + \frac{1}{2} D(b(x),\theta')\theta'' = 0.$$

Remark that $h(x)\theta'(x)\tanh(h(x)\theta'(x)) = \lambda h(x)$, which yields $h(x)\theta'(x) = F(\lambda h(x))$ or $h(x) = \frac{\tanh^{-1}(\frac{\lambda}{\theta'(x)})}{\theta'(x)}$. Deduce $b$ in function of $\theta'$ and $\lambda$ then

$$\frac{a_0'}{a_0} = -\frac{1}{2}\frac{D(b_\lambda(\theta'),\theta')}{E(b_\lambda(\theta'),\theta')}\theta''.$$

Explicit expression:

$$a_0(x) = [\ln(1-y^2)\tanh^{-1}y + y]^{-\frac{1}{2}}, y = \frac{\lambda}{\theta'(x)}.$$

# Action of the DTN on the GO formal approximation

## Proposition

*Let $\theta'$ solution of the eikonal equation. For a being a function of b, of class $C^4$, expressed as $a(x) = \mathbf{a}(b(x))$,*

$$e^{-i\theta(x)}Op(A(b(x),k))(ae^{i\theta}) - A(b(x),\theta'(x))a(b(x))$$
$$= -\frac{1}{2}\theta''(x)A''(b(x),\theta'(x))a(b(x)) - A'(b(x),\theta'(x))\frac{d}{dx}(a(b(x))$$
$$+O(b''(x),(b'(x))^2).$$

Remark: the same result does not hold for $C$ because $Op(C)$ does not send $\mathcal{S}$ to $\mathcal{S}$ (exponential growth). **Slight modification:** consider $\tilde{C}(b(x),k) := \frac{C(b(x),k)}{\cosh h_0 k}$ or, with $\Psi(x,z) = Op(A)(\phi) + Op(\tilde{C})(\tilde{\psi})$. Above proposition valid for $\tilde{C}$.

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Final result

$$Op\left(\begin{pmatrix} A & \tilde{C} \\ G_0 - \lambda & \frac{k}{\cosh h_0 k} \end{pmatrix}\right)\left[\begin{pmatrix} a_0 \\ \tilde{b}_0 \end{pmatrix} e^{i\theta}\right].\begin{pmatrix} -\theta' \\ C \end{pmatrix} = O(b'' + (b')^2).$$

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# Conclusion and perspectives

- Geometrical optics-physical optics seem to work at frequency 1
- Remainder term controlled, at each order $N$, by $\Pi_{l=0}^{p}(b^{(l)})^{k_l}$, with $\sum lk_l = N+2$
- Definition of the intermediate regime from the behavior of $X \to X \tanh X$,
- Obtain the GO solution of the mild slope equation as the GO eigenvector at order 0,
- Compare the modified mild slope quation (Chamberlain-Porter, Chamberlain) with the PO eigenvector,
- Compare with numerics,
- Compare with sloshing problem (Levitin, Polterovich, or Nigam).

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## Proof.

The proof uses $R(u) = e^{i[\theta(x+u)-\theta(x)-\theta'(x)u-\frac{\theta''(x)}{2}u^2]}$,
$u^4 S(u) = a(x+u) - a(x) - a'(x)u - \frac{1}{2}a''(x)u^2 - \frac{1}{6}a'''(x)u^3$. One then has

$$e^{-i\theta(x)}a(x+u)e^{i\theta(x+u)} = R(u)[a(x) + ua'(x) + \frac{1}{2}a''(x)u^2 + \frac{1}{6}a'''(x)u^3 + u^4 S(u)]e^{i\theta'(x)u+i\frac{\theta''(x)}{2}u^2}.$$

The Fourier transform of
$u \to [a(x) + ua'(x) + \frac{1}{2}a''(x)u^2 + \frac{1}{6}a'''(x)u^3 + u^4 S(u)]e^{i\frac{\theta''(x)}{2}u^2}$ at $\theta'(x) + \eta$ is the sum of
$a(x) + \frac{\eta}{\theta''(x)} + [(\frac{\eta}{\theta''(x)})^2 - \frac{i}{\theta''(x)}]a''(x) + [(\frac{\eta}{\theta''(x)})^3 + \frac{3i\eta}{(\theta''(x))^2}]e^{-i\frac{\eta^2}{2\theta''(x)}}$
and of the Fourier transform of $u \to u^4 S(u)e^{i\theta'(x)u+i\frac{\theta''(x)}{2}u^2}$. □

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## Proof.

The action of the symbol $A(b(x), k)$ on $u \to P(u)e^{i\theta'(x)u + i\frac{\theta''(x)}{2}u^2}$ is then the action of the symbol $A(b(x), \theta'(x) + \eta)$ on $u \to P(u)e^{i\frac{\theta''(x)}{2}u^2}$. Use then $\eta^4 T(\eta) = A(b(x), \theta'(x)) + A'(b(x), \theta'(x))\eta + \frac{1}{2}A''(b(x), \theta'(x))\eta^2 - \frac{1}{6}A'''(b(x), \theta'(x))\eta^3$. The three first terms act as $\eta^j = (\frac{1}{i}\frac{d}{du})^j$ and this action is handled by the terms $A(b(x), \theta'(x))a(x) + iA'(b(x), \theta'(x))a'(x) + \frac{1}{2}i\theta''(x)A''(b(x), \theta'(x))a(x)$. One is thus left to study

$$\int \eta^j \mathcal{F}(R) \star \mathcal{F}(u^4 S(u)e^{i\frac{1}{2}\theta''(x)u^2})(\eta)d\eta,$$

$$\oint \oint R(u)S(u)T(\eta)\eta^4 u^4 e^{i\frac{\theta''(x)}{2}u^2 - i\eta u}dud\eta.$$

## Proof.

Observe that

$(\frac{\partial}{\partial\eta})^4(\mathcal{F}(S(u)R(u)e^{i\frac{\theta''(x)}{2}u^2})) = \oint u^4 S(u)R(u)e^{i\frac{\theta''(x)}{2}u^2-iu\eta}du$, and that

$\oint\oint \eta^4 T(\eta)u^4 S(u)R(u)e^{i\frac{\theta''(x)}{2}u^2-iu\eta+it\eta}\frac{dud\eta}{2\pi} = Op(\eta^4 T)(RSu^4 e^{i\frac{\theta''(x)}{2}u^2})(t)$.

By use of Fourier transform, the action of $\eta^4$ is then the application of $(i\frac{\partial}{\partial t})^4$, hence the latter term rewrites

$$(i\frac{\partial}{\partial t})^4[\oint T(\eta)[(\frac{1}{i}\frac{\partial}{\partial\eta})^4\mathcal{F}(RSe^{i\frac{\theta''(x)}{2}u^2})(t)].$$

This induces the result. Note that, to obtain exactly our result, we need to keep only derivatives of $a$ up to order 2 (using a different version of $S$), use $R-1 = O(\theta''') = O(b'')$, in order to obtain thanks to $\tilde{S}(u) = \frac{a(x+u)-a(x)-a'(x)u}{u^2} = O(b''+(b')^2)$ the good estimates in the right hand term.