Cancellation Integration Scheme for the Magnetic Boundary Integral Operator on Curved Elements and Application to the Accurate Computation of the Radar Cross Section

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Introduction

Context: Analysis of the scattering of an electromagnetic field by a complex object

- electrically large
- physical/geometrical complexities

Motivation: Accurate computation of the Radar Cross Section (RCS) using boundary integral equations

What is it usually done?

- Electric Field Integral Equation (EFIE) for PEC scatterer or PMCHWT for dielectric formulation
- discretization with a Galerkin method using the Raviart-Thomas (RT) basis functions of lowest order
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Approximation theory of EFIE [1, 2]:

$$\|Q(\mathbf{J}) - Q(\mathbf{J}_h)\| \leq Ch^3$$

- Q : linear form on $H_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma)$ (observable: RCS, far field, ...)
- $\mathbf{J}$: surfacic current solution of the continuous problem
- $\mathbf{J}_h$: surfacic current solution of the discrete problem
- h : mesh size of the scatterer Γ (supposed to be regular)

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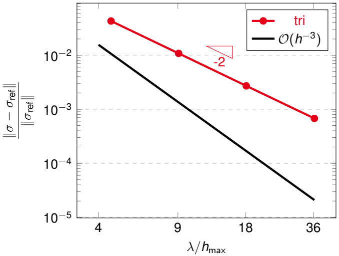

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How to improve the numerical convergence rate?

- ➥ increase the geometric approximation order as well as the polynomial one of the finite element space [3, 4, 5]
- important implementation within industrial codes

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Main difficulty: Accurate computation of elementary matrices

- ➥ singular integrals, no analytical formulas
- ➥ few satisfactory works in the literature [6, 7, 8, 9]
- ➥ new proposed cancellation integration scheme

Outline

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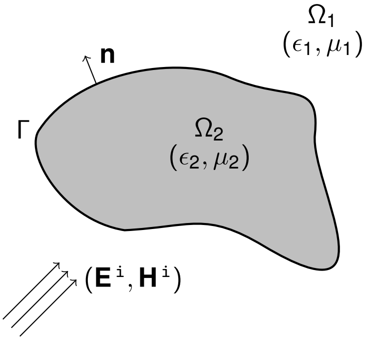
Model problem

Time-harmonic Maxwell's equations:

$$\begin{cases} \nabla \times \mathbf{E}_n + i k_n \eta_n \mathbf{H}_n = 0, \text{ in } \Omega_n, \quad n = 1,2 \\ \nabla \times \mathbf{H}_n - i \frac{k_n}{\eta_n} \mathbf{E}_n = 0, \text{ in } \Omega_n, \quad n = 1,2 \\ \mathbf{n} \times \mathbf{H}_1 = \mathbf{n} \times \mathbf{H}_2, \text{ on } \Gamma \\ \mathbf{n} \times \mathbf{E}_1 = \mathbf{n} \times \mathbf{E}_2, \text{ on } \Gamma \\ \text{Silver-Müller rad. cond. at infinity to } (\mathbf{E} - \mathbf{E}^{\mathrm{i}}, \mathbf{H} - \mathbf{H}^{\mathrm{i}}) \end{cases}$$

- total fields: $\mathbf{E}_n = \mathbf{E}_{|\Omega_n}, \quad \mathbf{H}_n = \mathbf{H}_{|\Omega_n}$
- $\omega = 2\pi f,$ f being the wave frequency
- $k_n = \omega\sqrt{\epsilon_n \mu_n}$ being the wavenumber
- $\eta_n = \sqrt{\frac{\mu_n}{\epsilon_n}},$ ϵ_n, μ_n being constants in Ω_n



Model problem

Let $\mathbf{J}_1 = \mathbf{n} \times \mathbf{H}_1$ and $\mathbf{M}_1 = \mathbf{n} \times \mathbf{E}_1$ be the electric and magnetic currents tangential to Γ

PMCHWT integral formulation [10]

$$\begin{cases} \mathbf{E}^{\mathtt{i}}_{\mathtt{tan}} = ik_2\eta_2 \mathrm{T}_{k_2}\mathbf{J}_1 + ik_1\eta_1 \mathrm{T}_{k_1}\mathbf{J}_1 + \mathrm{K}_{k_2}\mathbf{M}_1 + \mathrm{K}_{k_1}\mathbf{M}_1 \\ \mathbf{H}^{\mathtt{i}}_{\mathtt{tan}} = \frac{ik_2}{\eta_2}\mathrm{T}_{k_2}\mathbf{M}_1 + \frac{ik_1}{\eta_1}\mathrm{T}_{k_1}\mathbf{M}_1 - \mathrm{K}_{k_2}\mathbf{J}_1 - \mathrm{K}_{k_1}\mathbf{J}_1 \end{cases}, \quad \text{on } \Gamma$$

with the electric and magnetic boundary integral operators defined $\forall \mathbf{x} \in \Gamma$ by

$$\mathrm{T}_k\boldsymbol{\lambda}(\mathbf{x}) = \int_\Gamma G_k(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) + \frac{1}{k^2}\nabla_\Gamma \int_\Gamma G_k(\mathbf{x},\mathbf{y})\nabla_\Gamma \cdot \boldsymbol{\lambda}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})$$

$$\mathrm{K}_k\boldsymbol{\lambda}(\mathbf{x}) = \int_\Gamma \nabla_x G_k(\mathbf{x},\mathbf{y}) \times \boldsymbol{\lambda}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})$$

and $G_k(\mathbf{x},\mathbf{y}) = \dfrac{e^{-ik\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|}, \mathbf{x} \neq \mathbf{y}$

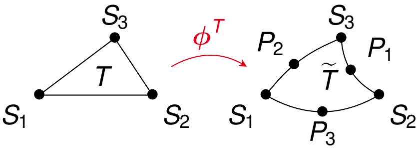
Outline

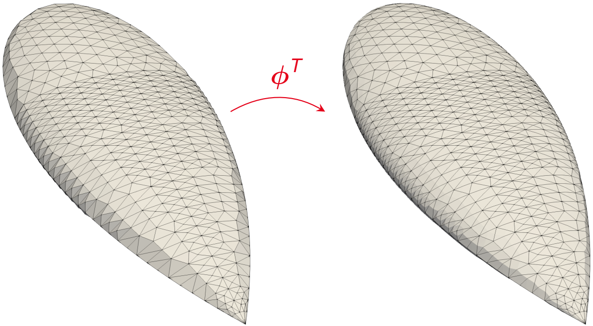
Quadratic approximation of the geometry

$$\phi^T : T \to \widetilde{T}$$

$$\phi^T(\mathbf{x}) = \mathbf{x} + \sum_{a=1}^{3} \mathbf{w}_a \lambda_{p_a}(\mathbf{x}) \lambda_{q_a}(\mathbf{x})$$

- $(a, p_a, q_a) \in \{(1,2,3), (2,3,1), (3,1,2)\}$
- $\lambda_{p_a}(\mathbf{x})$ being the area coord. $\mathbf{x}$ in T
- $\mathbf{w}_a = 4\left(\mathbf{P}_a - \frac{\mathbf{S}_{p_a}+\mathbf{S}_{q_a}}{2}\right)$, $\mathbf{P}_a$ being the projections of edge midpoints $[\mathbf{S}_{p_a}, \mathbf{S}_{q_a}]$ on Γ





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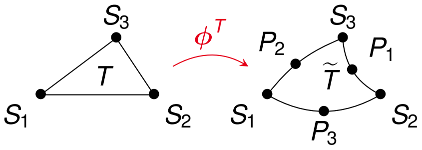
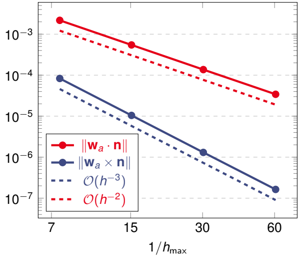


$$\|\mathbf{w}_a \cdot \mathbf{n}\| \leq Ch^2, \quad \|\mathbf{w}_a \times \mathbf{n}\| \leq Ch^3$$



➥ greater correction along the triangle normal

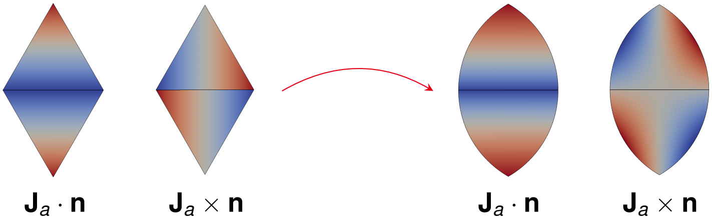
Curved Raviart-Thomas (RT) basis functions

RT finite element space of lowest order with basis functions [1, 11] defined $\forall a = 1, 2, 3$ by

$$\text{Flat RT:} \quad \mathbf{J}_a(\mathbf{x}) = \frac{\mathbf{x} - \mathbf{S}_a}{2|T|}, \qquad \text{Curved RT:} \quad \mathbf{J}_a(\mathbf{x}) = \frac{1}{|\mathcal{J}_{\phi^T}(\mathbf{x})|} \mathrm{d}\phi^T(\mathbf{x}) \frac{\mathbf{x} - \mathbf{S}_a}{2|T|}$$

- $\mathrm{d}\phi^T(\mathbf{x})$, $|\mathcal{J}_{\phi^T}(\mathbf{x})|$ being the differential, the jacobian of ϕ^T at $\mathbf{x}$ respectively, and $|T|$ being the area of T

Same properties as the flat RT basis functions:



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Cancellation integration scheme

Integral of the kind $\int_T \int_T = \frac{g(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^\alpha} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}, \quad \tilde{\mathbf{x}} = \phi^T(\mathbf{x}), \quad \tilde{\mathbf{y}} = \phi^T(\mathbf{y})$

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- change of integration variables to write $\mathbf{x} - \mathbf{y}$ as $\mathbf{x} - \mathbf{y} = u\gamma\mathcal{D}(\underbrace{u, \gamma, \omega, \tau}_{\text{new variables}})$

  to have $\tilde{\mathbf{x}} - \tilde{\mathbf{y}} = u\gamma\mathcal{R}(u, \gamma, \omega, \tau)$, $\mathcal{R} \neq 0$, $|Jac| = u\gamma\mathcal{J}(u, \gamma, \omega, \tau)$ and $g(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = (u\gamma)^{\alpha-1}\mathcal{G}(u, \gamma, \omega, \tau)$

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$$K_{ij} = \int_T\int_T G(R)\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|^3}\cdot(\mathbf{J}_i(\mathbf{y})\times\mathbf{J}_j(\mathbf{x}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}, \quad G(R) = (1+ikR)e^{-ikR}$$

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$$\begin{aligned} K_{ij} &= \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot (\mathbf{J}_i(\mathbf{y}) \times \mathbf{J}_j(\mathbf{x}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}, \quad G(R) = (1 + ikR)e^{-ikR} \\ &= \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot \Big((\mathbf{J}_i(\mathbf{y}) - \mathbf{J}_i(\mathbf{x})) \times \mathbf{J}_j(\mathbf{x})\Big)\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} + \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big)\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\ &= K_{ij}^1 + K_{ij}^2 \end{aligned}$$

Cancellation integration scheme

$$K_{ij}^{1}:\int_T\int_T G(R)\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|^3}\cdot\Big(\big(\mathbf{J}_i(\mathbf{y})-\mathbf{J}_i(\mathbf{x})\big)\times\mathbf{J}_j(\mathbf{x})\Big)\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}$$

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- Taylor expansion of $\mathbf{J}_i(\mathbf{x})$ around $\mathbf{y}$

$$\mathbf{J}_i(\mathbf{y}) - \mathbf{J}_i(\mathbf{x}) = -\mathrm{d}\mathbf{J}_i(\mathbf{y})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}\mathrm{d}^2\mathbf{J}_i(\mathbf{y})(\mathbf{x} - \mathbf{y})^2$$

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$$\mathbf{J}_i(\mathbf{y}) - \mathbf{J}_i(\mathbf{x}) = -\mathrm{d}\mathbf{J}_i(\mathbf{y})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}\mathrm{d}^2\mathbf{J}_i(\mathbf{y})(\mathbf{x} - \mathbf{y})^2$$

- Variables changes: $\mathbf{x} - \mathbf{y} = u\gamma\mathcal{D}(u, \gamma, \omega, \tau)$

$$\tilde{\mathbf{x}} - \tilde{\mathbf{y}} = u\gamma\mathcal{R}(u, \gamma, \omega, \tau), \quad \mathbf{J}_i(\mathbf{y}) - \mathbf{J}_i(\mathbf{x}) = u\gamma\mathcal{X}(u, \gamma, \omega, \tau), \quad |Jac| = u\gamma\mathcal{J}(u, \gamma, \omega, \tau)$$

➥ The triple product $|Jac|\left(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\right) \cdot \Big((\mathbf{J}_i(\mathbf{y}) - \mathbf{J}_i(\mathbf{x})) \times \mathbf{J}_j(\mathbf{x}) \Big)$ cancels the singularity $(u\gamma)^3$!

Cancellation integration scheme

$$K_{ij}^2 : \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x}) \Big) \mathrm{d}\mathbf{y} \mathrm{d}\mathbf{x}$$

Cancellation integration scheme

$$K_{ij}^2 : \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot \left(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\right) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}$$

- Taylor expansion of $\tilde{\mathbf{y}}$ around $\tilde{\mathbf{x}}$

$$\tilde{\mathbf{x}} - \tilde{\mathbf{y}} = \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}\mathrm{d}^2\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})^2$$

Cancellation integration scheme

$$K_{ij}^2 : \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}$$

- Taylor expansion of $\tilde{\mathbf{y}}$ around $\tilde{\mathbf{x}}$

$$\tilde{\mathbf{x}} - \tilde{\mathbf{y}} = \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}\mathrm{d}^2\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})^2$$

- Accounting for some geometric considerations
  - ➥ $\mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})$, $\mathbf{J}_i(\mathbf{x})$ and $\mathbf{J}_j(\mathbf{x})$ are in the same plane $\implies \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) \cdot \big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\big) = 0$

Cancellation integration scheme

$$K_{ij}^2 : \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}$$

- Taylor expansion of $\tilde{\mathbf{y}}$ around $\tilde{\mathbf{x}}$

$$\tilde{\mathbf{x}} - \tilde{\mathbf{y}} = \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}\mathrm{d}^2\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})^2$$

- Accounting for some geometric considerations
  - ➥ $\mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})$, $\mathbf{J}_i(\mathbf{x})$ and $\mathbf{J}_j(\mathbf{x})$ are in the same plane $\implies \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})) = 0$
  - ➥ $\Big(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\Big) \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) = -\frac{1}{2}\mathrm{d}^2\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})^2 \cdot (\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x}))$

Cancellation integration scheme

$$K_{ij}^2 : \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}$$

- Taylor expansion of $\tilde{\mathbf{y}}$ around $\tilde{\mathbf{x}}$

$$\tilde{\mathbf{x}} - \tilde{\mathbf{y}} = \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}\mathrm{d}^2\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})^2$$

- Accounting for some geometric considerations
  - ➥ $\mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})$, $\mathbf{J}_i(\mathbf{x})$ and $\mathbf{J}_j(\mathbf{x})$ are in the same plane $\implies \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})) = 0$
  - ➥ $\Big(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\Big) \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) = -\frac{1}{2}\mathrm{d}^2\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})^2 \cdot (\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x}))$
- Variables changes: $\mathbf{x} - \mathbf{y} = u\gamma\mathcal{D}(u, \gamma, \omega, \tau)$

$$\Big(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\Big) \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) = (u\gamma)^2\mathcal{Y}(u, \gamma, \omega, \tau), \quad |Jac| = u\gamma\mathcal{J}(u, \gamma, \omega, \tau)$$

Cancellation integration scheme

$$K_{ij}^2 : \int_T \int_T G(R) \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|^3} \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}$$

- Taylor expansion of $\tilde{\mathbf{y}}$ around $\tilde{\mathbf{x}}$

$$\tilde{\mathbf{x}} - \tilde{\mathbf{y}} = \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}\mathrm{d}^2\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})^2$$

- Accounting for some geometric considerations
  - ➥ $\mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})$, $\mathbf{J}_i(\mathbf{x})$ and $\mathbf{J}_j(\mathbf{x})$ are in the same plane $\implies \mathrm{d}\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})) = 0$
  - ➥ $\Big(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\Big) \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) = -\frac{1}{2}\mathrm{d}^2\phi^T(\mathbf{x})(\mathbf{x} - \mathbf{y})^2 \cdot (\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x}))$
- Variables changes: $\mathbf{x} - \mathbf{y} = u\gamma\mathcal{D}(u, \gamma, \omega, \tau)$

$$\Big(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\Big) \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big) = (u\gamma)^2\mathcal{Y}(u, \gamma, \omega, \tau), \quad |Jac| = u\gamma\mathcal{J}(u, \gamma, \omega, \tau)$$

➥ The product between the jacobian and $\Big(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\Big) \cdot \Big(\mathbf{J}_i(\mathbf{x}) \times \mathbf{J}_j(\mathbf{x})\Big)$ cancels the singularity $(u\gamma)^3$!

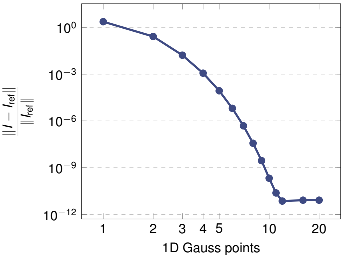
How to check the accuracy of the integration scheme?

Method: triangles refinement

- $K_i = \int_T F(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \mathbf{J}_i(\mathbf{y}) \mathrm{d}T(\mathbf{y})$ with $\tilde{\mathbf{y}} = \phi^T(\mathbf{y})$, $\mathbf{J}_i(\mathbf{y}) = \mathrm{d}\phi^T(\mathbf{y}) \frac{\mathbf{y} - \mathbf{S}_i}{2|T|}$
- T is meshed in small triangles $T = \bigcup_t \tau_t$
- On a small triangle τ_t : $\mathbf{x} - \mathbf{S}_i^T = \sum_{a=1}^{3} \lambda_a^\tau(\mathbf{S}_i^T)(\mathbf{x} - \mathbf{S}_i^\tau)$, then $K_i = \sum_t \int_{\tau_t} F(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \mathbf{J}_i(\mathbf{y}) \mathrm{d}\tau_t(\mathbf{y})$

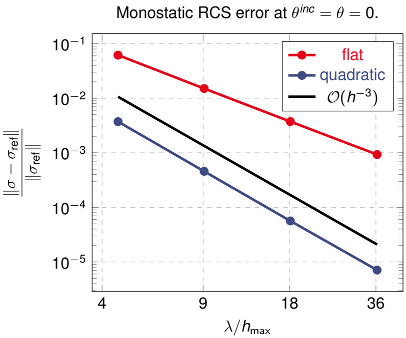
➥ computation of the integral with all possible configurations of pairs of triangles: disjoint triangles, triangles with one or two vertices in common and same triangles

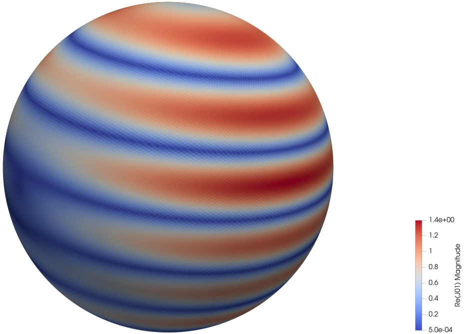
Quadrature error with respect to a 3-fold refined triangle.



Outline

Dielectric sphere of radius 1 m at 500 MHz, $\epsilon = 2$, $\mu = 0.5$





$|Re(\mathbf{J}_h)| \quad (A/m)$.

- Convergence rate in h^3 instead of h^2
- (Singular integral correctly computed!)

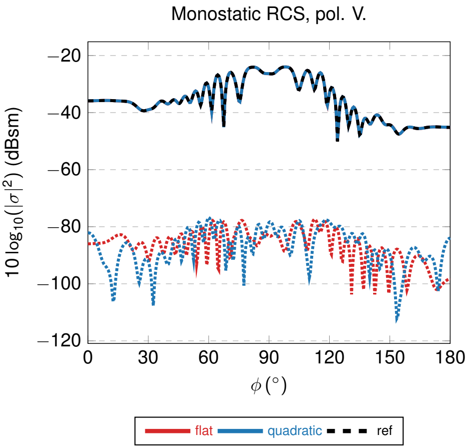
Dielecric NASA almond at 10.25 GHz, $\epsilon = 2$, $\mu = 0.5$

Configuration:

- l^{∞} relative error to reach: 10^{-3}
- reference flat mesh: 4 768 356 unknowns
- flat mesh: 545 760 unknowns
- curve mesh: 35 892 unknowns
- dense direct solver

Triangles	Assembly	Factorization	Solution	Total
Flat	125.70	223.70	11.71	**361.11**
Quadratic	0.38	0.03	0.01	**0.42**

CPU times (h·CPU).



CPU time savings by a factor of 860!

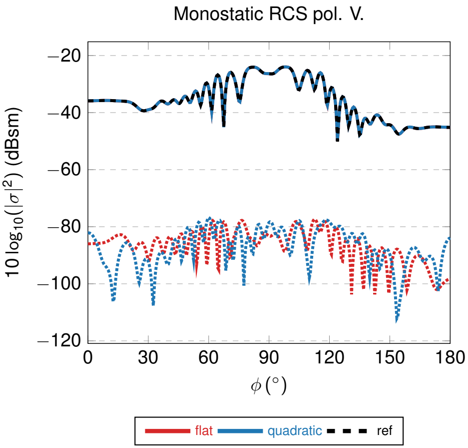
Dielecric NASA almond at 10.25 GHz, $\epsilon = 2$, $\mu = 0.5$

Configuration:

- l^∞ relative error to reach: 10^{-3}
- reference flat mesh: 4 768 356 unknowns
- flat mesh: 545 760 unknowns
- curve mesh: 35 892 unknowns
- $\mathcal{H}$-matrix solver

Triangles	Assembly	Factorization	Solution	Total
Flat	13.86	19.86	12.75	**46.47**
Quadratic	0.61	0.35	0.33	**1.29**

CPU times (min·CPU).

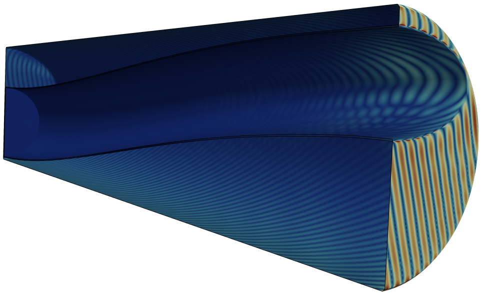


CPU time savings by a factor of 36!

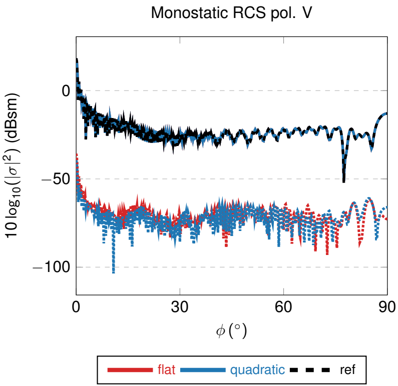
Worskshop ISAE 2018 : Channel cavity at 12 GHz

- coated cavity with dielectric of index ≈ 2 ($\epsilon = 1.5 - 0.1i$, $\mu = 2.5 - 1.8i$)
- reference flat mesh: 4 974 958 unknowns (x2 sym)
- flat and curve meshes: 1 879 015 unknowns (x2 sym)

➥ accuracy improved by a factor of 2 on both polarization: the geometry is not regular



$|Re(\mathbf{J}_h)| \quad (A/m).$

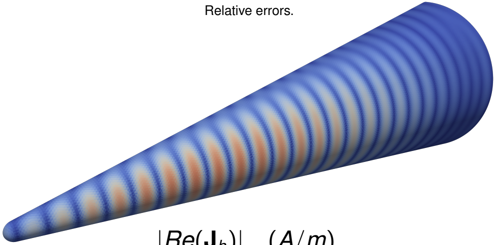


Dielectric coated metallic cone at 6 GHz

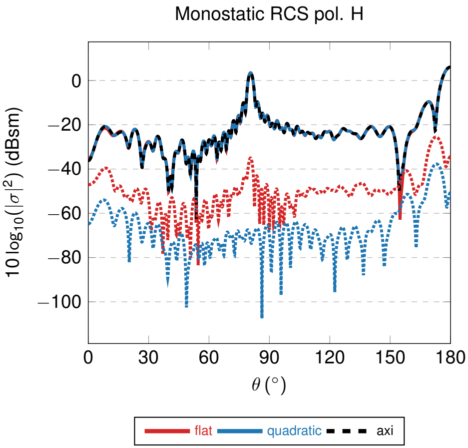
- dielectric index of 1.5 ($\epsilon = 2$, $\mu = 1.125$)
- flat and curve meshes: $\approx$ 317 500 unknowns

Triangles	l^∞ pol. H	l^∞ pol. V	l^2 pol. H	l^2 pol. V
flat	3e-2	1e-2	3e-2	2e-2
quadratic	6e-3	1e-3	7e-3	2e-3

Relative errors.

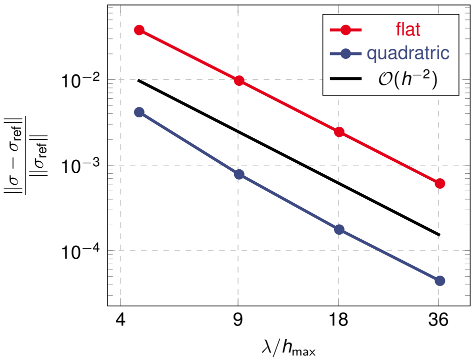


$|Re(\mathbf{J}_h)| \quad (A/m)$.

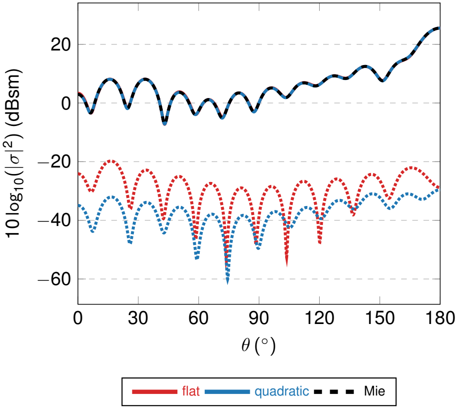


Impedant sphere of radius 1 m at 500 MHz, Z = 0.5

Monostatic RCS in $\theta^{inc} = \theta = 0$.

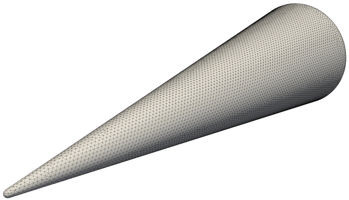


Bistatic RCS.



- Convergence rate in h^2 for curved and flat meshes:
  - ➥ solution in $L^2_{\mathtt{div}}(\Gamma)$ instead of $H^{-\frac{1}{2}}_{\mathtt{div}}(\Gamma)$
  - ➥ interpolation error in h instead of $h^{\frac{3}{2}}$

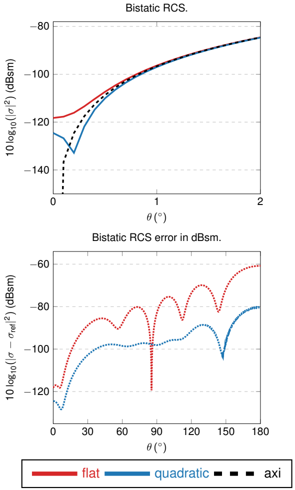
Cone at 1 GHz with impedance condition $Z = 1$



Weston's theorem [12]:

- If $Z = 1$ and the direction of incidence is along the revolution axis, then the backscattered field is equal to 0

➥ RCS computed with a curve mesh is 6 dB lower than with a flat mesh



Conclusion

In summary

- Bring up to date the quadratic elements for solving integral equations discretized with Raviart-Thomas basis functions
- Development of a new integration scheme for strongly-singular integrals with quadratic elements
- Proposition of a validation method to check the accuracy of integration schemes
- Numerical convergence order matches with the theoretical one
  - Accurate RCS for a given mesh
  - CPU time savings for a given accuracy

Future work

- Integration scheme for nearly singular integrals
- Using quadratic elements for FEM parts of a FEM-BEM coupling

Thanks for your attention! Any questions?

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